

General study and basic properties of causal symmetries

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Abstract. We fully develop the concept of causal symmetry introduced in [21]. A causal symmetry is a transformation of a Lorentzian manifold (V, \mathbf{g}) which maps every future-directed vector onto a future-directed vector.

We prove that the set of all causal symmetries is not a group under the usual composition operation but a *submonoid* of the diffeomorphism group of V . Therefore, the infinitesimal generating vector fields of causal symmetries — *causal-preserving vector fields*— are associated to local one-parameter groups of transformations which are causal symmetries only for positive values of the parameter —*one-parameter submonoids of causal symmetries*—. The pull-back of the metric under each causal symmetry results in a new rank-2 future tensor, and we prove that there is always a set of null directions canonical to the causal symmetry. As a result of this it makes sense to classify causal symmetries according to the number of independent canonical null directions. This classification is maintained at the infinitesimal level where we find the necessary and sufficient conditions for a vector field to be causal preserving. They involve the Lie derivatives of the metric tensor and of the canonical null directions. In addition, we prove a stability property of these equations under the repeated application of the Lie operator. Monotonicity properties, constants of motion and conserved currents can be defined or built using casual preserving vector fields. Some illustrative examples are presented.

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1. Introduction

Symmetry is one of the truly important concepts in Physics. For gravitational theories, such as General Relativity and the like, symmetries have always been taken into account in many fruitful ways, in particular in order to analyse and understand the solutions to Einstein's field equations [51] with groups of motions (infinitesimal isometries). This analysis has been partly generalized considering more general symmetries, such as homothetic or conformal Killing vectors. Nevertheless, as far as we know, the question of whether there are vector fields which, in some sense, preserve the causal structure of the spacetime has not been considered in full generality. This is our aim in this paper. To that end, we will base our study on the concept of isocausality introduced in [20]. We will encounter several mathematical problems to be faced, such as the generic infinite-dimensionality of the vector space we define, and the absence of a truly Lie algebra, or Lie group, structure. These impediments are briefly explained in this introduction.

Symmetries on a spacetime are often described by a group G acting on the underlying manifold and such that some adequate property—the metric, the null cone, the connection, etcetera— is preserved by this action. Therefore every action gives rise to a group of transformations of the manifold V which is a subgroup of the diffeomorphism group. Usually, all the essential information of the symmetry group is contained in the so-called *infinitesimal generators* of the group (defined in a way we briefly review in section 3) which are vector fields on the manifold obeying an appropriate differential condition —generally constraining the Lie derivative of certain geometric objects along these vector fields—. Obvious examples are the Killing and conformal Killing vectors for isometries or conformal motions, respectively. Quite frequently, this set of vector fields form a Lie algebra under the Lie bracket operation $[\cdot, \cdot]$. Its dimension, however, can be finite or not:

Finite dimensional Lie algebra. In this case the Lie algebra is isomorphic to the intrinsic Lie algebra of a finite-dimensional Lie group G , the transformation group being a realization of G over the manifold. All this is well known and can be found in many textbooks (see e. g. [43, 54, 14, 51]). Actually, finite dimensional Lie groups have been studied for more than one hundred years and their main features as well as the general theory are already widely known. Thus, it is possible to use all the machinery of Lie group theory as it has been largely done for example with isometries in order to find, classify and study solutions of four-dimensional Einstein’s field equations [51]. Similar analyses have been performed for conformal motions, affine collineations, linear collineations and conformal collineations (see [54, 2, 26, 27, 28, 15, 29, 51] and references therein) with perhaps a less firmly established physical interpretation. An account of the classical symmetries studied in General Relativity can be found in [38, 51].

Infinite dimensional Lie algebra. Some examples of such Lie algebras are provided, in particular situations, by the generators of Riemann tensor collineations [23] or of 2-dimensional conformal motions. The infinite-dimensional case is still an open subject since there is no known general scheme to deal with it, hence, it remains highly unexplored apart from isolated attempts [53, 55, 34, 16]. Nevertheless, we quote a first classification of infinite-dimensional Lie algebras presented in [37]:

- (i) Case of vector fields on differentiable manifolds. This is the typical case in General Relativity. The main result is the classification of all simple Lie algebras performed by Cartan [12].
- (ii) Smooth mappings from differentiable manifolds onto finite dimensional Lie groups. Since Lie groups can be realized as groups of matrices, here we deal with matrices whose elements are smooth functions. An outstanding example is the group of rigid motions of \mathbb{R}^n ([4] and references therein) where the manifold is \mathbb{R} and the Lie group is the semidirect product of $SO(n)$ with \mathbb{R}^n .
- (iii) Linear operators in Hilbert spaces, the common case in Quantum Mechanics.
- (iv) Kac-Mody algebras.

We will be concerned with (i), groups acting on differentiable manifolds. To get a flavour of the differences between finite and infinite-dimensional Lie algebras, recall that for a finite-dimensional Lie group G , the orbits of the action are differentiable submanifolds of the manifold, and sometimes one can construct these orbits explicitly. Furthermore, the generating vector fields are smooth complete vector fields. This result has a converse (Palais’ theorem, see [44]). Contrarily, if the group G is infinite dimensional, the previous nice properties no longer hold, see [25] for a summary.

This work is an extension and a development of the ideas put forward in [21] where we defined a symmetry transformation (called causal symmetry) which generalizes the well known case of conformal symmetries. The basic requisite for a transformation to be a causal symmetry is that all future-directed vectors must be push-forwarded to future-directed vectors, so causal symmetries are simply causal mappings (see [20] for an explanation of this terminology) in which both the domain and the target manifolds are the same. A similar transformation was used in [31, 30] with other aims. These authors use the name “causally decreasing diffeomorphisms” for our causal symmetries and they study some of their properties under certain restrictions. We address the subject in its full generality —recovering results in [31, 30]— and this is the reason why we have preferred to keep our terminology and notation. Nonetheless we will indicate which of our results were already presented in those references.

The main difference between causal symmetries and the above outlined more typical transformations is that the former do not form a group. Rather, they constitute a *monoid* of transformations which means that the inverse of a causal symmetry is not in general a causal symmetry. As a matter of fact, the only causal symmetries whose inverse is also a causal symmetry are the conformal transformations, which are thus included and perfectly identifiable within the general set of causal symmetries. Nevertheless, the set of causal symmetries is a *submonoid* of the diffeomorphism group of the manifold, or of one of its subgroups, so we can still apply some of the techniques and results used when dealing with groups of transformations.

Submonoids and their generalizations, called semigroups, are far from new structures and constitute a branch of research on their own (the difference between submonoids and semigroups is that semigroups have no identity element). A thorough study of these structures can be found in [35, 36, 41]. They play, for instance, a key role in the study of one-parameter semigroups of linear operators in Banach spaces (see e.g. [17]). There are also examples of the relevance of submonoids of transformations in physics. For instance, in Quantum Mechanics the general time evolution of a state or a density matrix is ruled, under certain assumptions, by the so-called dynamical maps. The inverse of a dynamical map is not a dynamical map —unless the map transforms pure states in pure states—. This is quite logical since there can be no evolution from a mixed state, which is always the outcome of a measure, to a pure state, which are states existing before any measure takes place, see [40]. In an analogous manner, the causal symmetries we define, and their infinitesimal counterpart, will select or prefer a direction of time, the “future”, making it thus impossible that the inverse transformations—necessarily leading to the “past”— be causal symmetries. This seems reasonable from a physical point of view, but involves the previously mentioned mathematical difficulties: submonoid structures and infinite dimensionality of the generating vector spaces.

Most of the material presented in this paper relies in the research carried out in [10] and [20] (henceforth PI and PII respectively). A summary of them together with some new results dealing with causal tensors have been placed in appendix A which should be first studied by the reader not acquainted with these matters. We will refer either to this appendix or to any of those references when some of the results stated there is required. Part of our notation is also defined in this appendix although most of it was already used in PI or PII.

The plan of the paper is as follows: in section 2 we introduce causal symmetries, their principal null directions, and some of their basic properties. Section 3 is devoted to the study of one-parameter submonoids of causal symmetries, their classification in

terms of canonical null directions and the definition of invariant subsets of the manifold with the same number of null directions. The necessary and sufficient conditions which the generating vector fields of causal symmetries must meet are found in section 4 giving the most general equation involving the Lie derivative of the metric tensor with respect to these generators. Finally, constants of motions and currents are constructed for causal symmetries in section 5. Some examples are provided throughout. We close the paper with some conclusions.

2. Causal symmetries

Let us first of all set the notation used in this work. V will stand for a differentiable Lorentzian n -dimensional manifold with appropriate smooth structures and metric g_{ab} (the signature convention is $(+, -, \dots, -)$). The further condition of analyticity will be required for some results. Round and square brackets enclosing indexes denote symmetrization and antisymmetrization respectively. The tangent space at a point $x \in V$, $T_x(V)$, gives rise to the tangent bundle $T(V)$ and cotangent bundle $T^*(V)$ in the standard fashion as well as the bundle $T_s^r(V)$ of r -contravariant and s -covariant tensors formed from these two by means of the tensor product \otimes . Boldface arrowed (un-arrowed) characters will represent contravariant (covariant) objects reserving the small letters for the specific case of vectors and 1-forms. We make no distinction neither in our notation nor in our terminology between the elements of the previously cited bundles and their sections unless otherwise stated. In these cases, the value of the section \vec{u} at the point x shall be denoted by $\vec{u}|_x$. Vectors and 1-forms are related by the usual rule $\mathbf{w} = \mathbf{g}(\cdot, \vec{w})$ written in index notation as $w_a = g_{ab}w^b$ so we can translate all definitions given for contravariant objects into a covariant form and vice-versa (we will always use the same kernel letter for a vector and its associate 1-form). By means of a suitable raising or lowering of indices, every rank-2 tensor can be brought into an element of $T_1^1(V)$ which can be regarded as an endomorphism on $T(V)$. We can do this as T_b^a and T_a^b for T_{ab} being both endomorphisms equivalent in the symmetric case $T_{ab} = T_{ba}$. Therefore when we speak about the eigenvectors and eigenvalues of any rank-2 tensor we will mean those of the mixed 1-1 tensor T_b^a .

For a diffeomorphism $\varphi : V \rightarrow W$ the push-forward and pull-back are written as φ' and φ^* respectively. For later use we recall the following formula involving these operations: if $u^a(x)$ and $v_a(x)$ are differentiable sections of $T(V)$ and $T^*(V)$ respectively expressed in a local coordinate chart, and $y = \varphi(x)$ a diffeomorphism of V in these coordinates, then the pull-back of the scalar $k(x) = v_a(x)u^a(x)$ is

$$(\varphi^*k)(x) = k(\varphi(x)) = (\varphi^*v)_b(x)(\varphi'^{-1}u)^b(x) \quad (2.1)$$

with the obvious generalization for higher rank tensors. The smooth diffeomorphisms acting on V form an infinite dimensional continuous group denoted by $Diff(V)$.

We present next the main objects in this work. These are causal relations in which both the domain manifold and the target manifold are the same Lorentzian manifold (V, \mathbf{g}) (see appendix A for notation and explanations).

Definition 2.1 For a Lorentzian manifold (V, \mathbf{g}) we define $\mathcal{C}(V, \mathbf{g})$ as the set of diffeomorphisms φ such that $V \prec_\varphi V$. The elements of $\mathcal{C}(V, \mathbf{g})$ are called causal symmetries.

Remark. As shown in appendix A a diffeomorphism $\varphi \in \mathcal{C}(V, \mathbf{g})$ iff $\varphi^*\mathbf{g} \in \mathcal{DP}_2^+(V)$ (theorem A.2) or, equivalently, iff $\varphi^*\mathbf{g}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \geq 0$ for all $\vec{\mathbf{k}} \in \partial\Theta_x$, $\forall x \in V$ (theorem A.3).

According to the comments of appendix A, the set $\mathcal{C}(V, \mathbf{g})$ contains as a proper subset the group of conformal symmetries of our manifold which we denote by $\text{Conf}(V, \mathbf{g})$. The general properties of causal relations as well as the relationship between causal and conformal relations discussed in appendix A translates also to the set of causal symmetries as follows (see proposition A.7 for the definition of $\mu(\varphi)|_x$):

Proposition 2.1 *The next properties hold*

- (i) $\varphi_1, \varphi_2 \in \mathcal{C}(V, \mathbf{g}) \Rightarrow \varphi_1 \circ \varphi_2 \in \mathcal{C}(V, \mathbf{g})$.
- (ii) $\text{Conf}(V, \mathbf{g}) = \mathcal{C}(V, \mathbf{g}) \cap \mathcal{C}(V, \mathbf{g})^{-1}$.
- (iii) $\varphi_1, \varphi_2 \in \mathcal{C}(V, \mathbf{g}) \Rightarrow \mu(\varphi_1 \circ \varphi_2)|_x \subseteq \mu(\varphi_2)|_x$.

Proof : Properties *i*) and *ii*) are straightforward consequences of the analog properties for causal relations (proposition 3.3 of PII) and so we will not repeat the proof here. Property *iii*) follows from the equation

$$0 = (\varphi_2^* \varphi_1^* \mathbf{g})|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = (\varphi_1^* \mathbf{g})|_{\varphi_2(x)}(\varphi_2' \vec{\mathbf{k}}, \varphi_2' \vec{\mathbf{k}}) \quad \forall \vec{\mathbf{k}} \in \mu(\varphi_1 \circ \varphi_2)|_x,$$

because the null vector $\vec{\mathbf{k}}$ remains null under φ_2' (see proposition A.7). \square

This proposition tells us that $\mathcal{C}(V, \mathbf{g})$ with the inner operation of diffeomorphism composition is a submonoid of $\text{Diff}(V)$. Recall that a set S with an inner operation is a monoid if the operation is associative and there exists an identity element for this operation in S (if such identity does not exist S is called a semigroup). Furthermore, if $S \subset G$ is a *proper submonoid* of G we define its group of units $H(S)$ as $S \cap S^{-1}$. The previous proposition tells us what is the group of units of $\mathcal{C}(V, \mathbf{g})$.

Proposition 2.2 *The group of units of $\mathcal{C}(V, \mathbf{g})$ is $\text{Conf}(V, \mathbf{g})$.* \square

An interesting property of the group of units is that it is the maximal subgroup contained in S in the sense that there is no other bigger subgroup of G contained in S possessing $H(S)$ as a proper subgroup. See [35] for the proof of this and other properties of submonoids and semigroups. The causal symmetries not being conformal transformations deserve thus a special name.

Definition 2.2 *Any $\varphi \in \mathcal{C}(V, \mathbf{g}) \setminus \text{Conf}(V, \mathbf{g})$ is called a proper causal symmetry.*

As is clear $\mathcal{C}(V, \mathbf{g})$ depends on the background metric \mathbf{g} chosen for our Lorentzian manifold. Nonetheless some properties of $\mathcal{C}(V, \mathbf{g})$ are shared by different elections of the metric \mathbf{g} , in particular $\mathcal{C}(V, \mathbf{g})$ is conformally invariant:

Proposition 2.3 $\mathcal{C}(V, \mathbf{g}) = \mathcal{C}(V, \sigma \mathbf{g})$ for any positive smooth function σ on V .

Proof : This happens because a future tensor with respect to a fixed background metric \mathbf{g} will remain future with respect to every conformal metric $\sigma \mathbf{g}$ if σ is positive, and this applies to $\varphi^* \mathbf{g}$, $\forall \varphi \in \mathcal{C}(V, \mathbf{g})$. \square

Another result connecting the causal structure of (V, \mathbf{g}) as it was defined in PII and the set of causal symmetries is the next (see definition A.5 for the concept of *isocausality*)

Proposition 2.4 *For two isocausal Lorentzian manifolds (V, \mathbf{g}) and $(W, \tilde{\mathbf{g}})$, there is a one-to-one correspondence between the sets $\mathcal{C}(V, \mathbf{g})$ and $\mathcal{C}(W, \tilde{\mathbf{g}})$.*

Remark. Note that this is a nontrivial result as the cardinality of the diffeomorphism group of each manifold need not be the same as the cardinality of $\mathcal{C}(V, \mathbf{g})$ or $\mathcal{C}(W, \tilde{\mathbf{g}})$.

Proof : Under the hypotheses of this proposition we know that $V \prec_{\psi_1} W$ and $W \prec_{\psi_2} V$. Now, it is rather simple to check that for every $\varphi \in \mathcal{C}(V, \mathbf{g})$, $\psi_1 \circ \varphi \circ \psi_2 \in \mathcal{C}(W, \tilde{\mathbf{g}})$. Similarly every $\chi \in \mathcal{C}(W, \tilde{\mathbf{g}})$ gives rise to an element of $\mathcal{C}(V, \mathbf{g})$ defined as $\psi_2 \circ \chi \circ \psi_1$. These correspondences are injective so that Bernstein equivalence theorem [32] leads to the result. \square

If we consider now elements of $\mathcal{DP}_r^+(V)$ (see appendix A, definition A.1) we know according to proposition A.6 that they will remain in $\mathcal{DP}_r^+(V)$ under the pull-back of every diffeomorphism of $\mathcal{C}(V, \mathbf{g})$. The next proposition tells us that their principal directions also behave in a precise fashion when push-forwarded by the elements of $\mathcal{C}(V, \mathbf{g})$ (see definition A.3 for the set $\sigma(\mathbf{T})$).

Proposition 2.5 *If $\varphi \in \mathcal{C}(V, \mathbf{g})$ then we have the following inclusions $\forall x \in V$:*

- (i) $\varphi'[\mu(\varphi^*\mathbf{T}|_x)] \subseteq \mu(\mathbf{T}|_{\varphi(x)}) \forall \mathbf{T} \in \mathcal{DP}_r^+(V)$.
- (ii) $\mu(\varphi^*\mathbf{T}|_x) \subseteq \mu(\varphi)|_x \forall \mathbf{T} \in \mathcal{DP}_r^+(V)$.
- (iii) $\forall \mathbf{T} \in \mathcal{DP}_2^+(V)$, either
 - (a) $\mathbf{T}_1 \times_1 \mathbf{T} = 0 \implies \sigma(\varphi^*\mathbf{T}|_x) = \sigma(\mathbf{T}|_x) = \partial\Theta_x^+$,
 - (b) or $\sigma(\varphi^*\mathbf{T}|_x) \subseteq \mu(\varphi)|_x$ and $\varphi'[\sigma(\varphi^*\mathbf{T}|_x)] \subseteq \sigma(\mathbf{T}|_{\varphi(x)})$.

Proof : The first two points follow from

$$0 = \varphi^*\mathbf{T}|_x(\vec{\mathbf{k}}, \dots, \vec{\mathbf{k}}) = \mathbf{T}|_{\varphi(x)}(\varphi'\vec{\mathbf{k}}, \dots, \varphi'\vec{\mathbf{k}}),$$

where $\vec{\mathbf{k}} \in \mu(\varphi^*\mathbf{T}|_x)$ (note that $\varphi'\vec{\mathbf{k}}|_{\varphi(x)}$ is then null for every $\vec{\mathbf{k}}|_x \in \mu(\mathbf{T}|_x)$). To prove (iii), from the comments after definition A.3 one knows that every $\vec{\mathbf{k}}_1 \in \sigma(\varphi^*\mathbf{T}|_x)$ is either in $\mu(\varphi^*\mathbf{T}|_x)$ or, otherwise, there is another null $\vec{\mathbf{k}}_2$ (given by $k_2^b = T^b_c k_1^c$) such that

$$0 = \varphi^*\mathbf{T}|_x(\vec{\mathbf{k}}_2, \vec{\mathbf{k}}_1) = \mathbf{T}|_{\varphi(x)}(\varphi'\vec{\mathbf{k}}_2, \varphi'\vec{\mathbf{k}}_1). \quad (2.2)$$

In the former case the statement is a subcase of point (i). Suppose then that $\vec{\mathbf{k}}_1 \notin \mu(\varphi^*\mathbf{T}|_x)$. From (2.2) we deduce, \mathbf{T} being a future tensor, that at least one of $\varphi'\vec{\mathbf{k}}_1|_{\varphi(x)}$ or $\varphi'\vec{\mathbf{k}}_2|_{\varphi(x)}$ must be null. If both of them are null, (b) is proven. If $\varphi'\vec{\mathbf{k}}_1|_{\varphi(x)}$ were null but $\varphi'\vec{\mathbf{k}}_2|_{\varphi(x)}$ were timelike, (2.2) would imply that in fact $\mathbf{T}|_{\varphi(x)}(\cdot, \varphi'\vec{\mathbf{k}}_1) = 0$ so that $\varphi^*\mathbf{T}|_x(\cdot, \vec{\mathbf{k}}_1) = 0$ and thus $\vec{\mathbf{k}}_1$ would be in $\mu(\varphi^*\mathbf{T}|_x)$ against the assumption. Finally, if $\varphi'\vec{\mathbf{k}}_2|_{\varphi(x)}$ is null but $\varphi'\vec{\mathbf{k}}_1|_{\varphi(x)}$ is timelike then, as before, $\varphi^*\mathbf{T}|_x(\vec{\mathbf{k}}_2, \cdot) = 0$ whence $\vec{\mathbf{k}}_2 \in \mu(\varphi^*\mathbf{T}|_x) \subseteq \mu(\varphi)|_x$ and, furthermore, $\varphi^*\mathbf{T}|_x = \mathbf{k}_2 \otimes \mathbf{v}$ for some $\mathbf{v} \in \mathcal{DP}_1^+|_x$. It follows that $\mathbf{T} = \mathbf{k}_2 \otimes \mathbf{u}$ for a future-directed causal \mathbf{u} , in which case, as is clear, $\mathbf{T}_1 \times_1 \mathbf{T} = 0$ and every null vector belongs to $\sigma(\varphi^*\mathbf{T}|_x) = \sigma(\mathbf{T}|_x)$. \square

A first consequence of this result is that $\mu(\varphi^*\mathbf{T}|_x)$ will be empty if either $\mu(\varphi)|_x$ itself is empty or $\mu(\mathbf{T}|_{\varphi(x)}) = \emptyset$. Similar considerations hold for $\sigma(\varphi^*\mathbf{T}|_x)$. Unless for the particular case of tensors $\mathbf{T}_1 \times_1 \mathbf{T} = 0$, part of the above results can be gathered in the chain $\mu(\varphi^*\mathbf{T}|_x) \subseteq \sigma(\varphi^*\mathbf{T}|_x) \subseteq \mu(\varphi)|_x$, which is a property to be used later on, and that can be refined if we impose further algebraic constraints upon \mathbf{T} when it has rank 2.

Proposition 2.6 *If the symmetric $\mathbf{T} \in \mathcal{DP}_2^+(V)$ has Lorentzian signature at every point of the manifold then $\sigma(\varphi^*\mathbf{T}|_x) = \mu(\varphi^*\mathbf{T}|_x)$, $\forall x \in V$ and every $\varphi \in \mathcal{C}(V, \mathbf{g})$.*

Proof : The signature of $\varphi^*\mathbf{T}|_x$ is the same as the signature of $\mathbf{T}|_x$ so the result follows from application of proposition A.5 to the tensor $\varphi^*\mathbf{T}|_x$. \square

An immediate consequence of this is

$$\sigma(\varphi^*\mathbf{g}) = \mu(\varphi^*\mathbf{g}) = \mu(\varphi), \quad \forall \varphi \in \mathcal{C}(V, \mathbf{g}).$$

The tensor $\varphi^*\mathbf{g}|_x$ may change its algebraic character from point to point so the number of independent canonical null directions will in general be different depending on the point of V . Therefore, it is quite reasonable to collect in sets the points of the manifold with the same number of these directions.

Definition 2.3 Let φ be an element of $\mathcal{C}(V, \mathbf{g})$ and consider the future tensor $\varphi^*\mathbf{g}$. Then we define the following sets:

$$\begin{aligned} \mathcal{N}_\varphi^m &= \{x \in V : \varphi^*\mathbf{g}|_x \text{ has } m \text{ linearly independent null eigenvectors}\}, \\ \mathcal{U}_\varphi &= \{x \in V : \mu(\varphi^*\mathbf{g}|_x) = \emptyset\}. \end{aligned}$$

The union of the sets \mathcal{N}_φ^m for all the values of m is denoted by \mathcal{N}_φ .

The set \mathcal{N}_φ^m is formed by the points $x \in V$ at which $(\varphi^*\mathbf{g})|_x$ has Segre type $\overbrace{[(1, 1 \dots 1) 1 \dots 1]}^m$ and its spatial degeneracies (or $[2 1 \dots 1]$ and its degeneracies if $m = 1$). The allowed Segre types for a rank-2 causal tensor are summarized in table A1 of appendix A.

Following [50], whenever we work with a smooth section of $\mathcal{DP}_2^+(V)$, we can decompose our spacetime V disjointly in sets defined by having a constant Segre type of the section values at their interior, plus a remainder X with no interior which is closed in V . This is an application of theorem 1 of [50] which states that a section of the bundle $T_2^0(V)$ of symmetric rank-2 tensors on a four-dimensional Lorentzian manifold gives rise to a decomposition of V in the following disjoint subsets:

$$\begin{aligned} V = X \cup [1, 111] \cup [1, 1(11)]^\circ \cup [(1, 1)11]^\circ \cup [(1, 1)(11)]^\circ \cup [(1, 11)1]^\circ \cup [1, (111)]^\circ \cup \\ [(1, 111)]^\circ \cup [211]^\circ \cup [2(11)]^\circ \cup [(21)1]^\circ \cup [(211)]^\circ \cup [31]^\circ \cup [(31)]^\circ \cup [z\bar{z}11] \cup [z\bar{z}(11)]^\circ, \end{aligned}$$

where the Segre notation is also used to mean each of the sets whose interior has constant algebraic type. If the section is a causal tensor then we must only take into account its allowed Segre types, shown in table A1. For the section $\varphi^*\mathbf{g}$ we thus have that these sets are related to \mathcal{U}_φ and \mathcal{N}_φ^m in the following way for dimension four:

$$\begin{aligned} \mathcal{U}_\varphi &= [1, 111] \cup [1, 1(11)] \cup [1, (111)], \quad \mathcal{N}_\varphi^1 = [211] \cup [2(11)] \cup [(21)1] \cup [(211)], \\ \mathcal{N}_\varphi^2 &= [(1, 1)11] \cup [(1, 1)(11)], \quad \mathcal{N}_\varphi^3 = [(1, 11)1], \quad \mathcal{N}_\varphi^4 = [(1, 111)]. \end{aligned}$$

Thus the splitting of V in terms of these sets looks like:

$$V = \mathcal{U}_\varphi \cup (\mathcal{N}_\varphi^1)^\circ \cup (\mathcal{N}_\varphi^2)^\circ \cup (\mathcal{N}_\varphi^3)^\circ \cup (\mathcal{N}_\varphi^4)^\circ \cup X. \quad (2.3)$$

Of course this splitting depends on the element φ of $\mathcal{C}(V, \mathbf{g})$ under consideration and we may have different splittings of our manifold for the different elements of $\mathcal{C}(V, \mathbf{g})$. This is of no relevance here but it will play a role when we study continuous causal symmetries in the next section where we will show that the resulting sets defined by means of these splitting techniques have nice invariant properties. We must also add that, as the authors of [50] argue at the end of their paper, their result can also be extended to dimensions greater than four so we will assume henceforth the obvious n -dimensional generalization of (2.3).

3. Continuous theory

As briefly explained in the Introduction, when one studies actions of symmetry groups on a spacetime such as isometries or conformal motions, one uses the intimate relationship between the group of transformations itself and its Lie algebra of “infinitesimal generators”. This can be done in fact not only for the finite dimensional cases, but also for the Lie algebras of infinite dimension. In any case, one goes from the global theory to the local one by introducing local one-parameter groups of diffeomorphisms belonging to the transformation group under consideration. These local one-parameter groups of transformations are denoted by $\{\varphi_s\}_{s \in I}$ where s is the canonical parameter, I is an interval of the real line containing 0 and $\varphi_0 = Id$. The possibility of working with global one-parameter groups with $I = \mathbb{R}$ or S^1 depends on the group under consideration although this is irrelevant to our purposes. A vector field $\vec{\xi}$ is said to be an infinitesimal generator of the symmetry group (a *generating vector* in short) if there exists one of the above defined one-parameter groups $\{\varphi_s\}_{s \in I}$ such that in local coordinates $\{x^a\}$:

$$\xi^a(x) = \left. \frac{d\varphi_s^a(x)}{ds} \right|_{s=0} \quad \forall x \in V.$$

The set of generators form a Lie algebra under the usual Lie bracket of vector fields. Sometimes it is possible to find a differential condition which determines all the generating vectors of a certain type of symmetry as for instance $\mathcal{L}_{\vec{\xi}} \mathbf{g} = 0$ or $\mathcal{L}_{\vec{\xi}} \mathbf{g} = \alpha \mathbf{g}$ for isometries and conformal motions, respectively. Therefore, the knowledge of such differential conditions may provide full information about the symmetry under study, so its search for the causal symmetries is one of our main tasks from now on.

Nevertheless, an immediate difficulty arises: when dealing with the causal symmetries we are confronted with the problem that $\mathcal{C}(V, \mathbf{g})$ is not actually a group. However, $\mathcal{C}(V, \mathbf{g})$ keeps group-like structures, hence we may expect that some of the above mentioned properties of other symmetries will also be maintained for the causal symmetries. To start with, we already know that $\mathcal{C}(V, \mathbf{g})$ has the algebraic structure of a monoid and this allows us to say a word about any topological group with a subset liable to be realized by means of elements of $\mathcal{C}(V, \mathbf{g})$. Of course, we will be mainly concerned with the nontrivial case where those elements are proper causal symmetries so that $\mathcal{C}(V, \mathbf{g}) \setminus Conf(V, \mathbf{g})$ should be nonempty. In this case, if G is the mentioned topological group and $S \subset G$ is the set which gives rise to $\mathcal{C}(V, \mathbf{g})$ then S cannot be a compact subset of G (see [35], pag 370). Moreover, and despite $\mathcal{C}(V, \mathbf{g})$ not being a group, we can still define one-parameter groups $\{\varphi_s\}_{s \in I}$ such that they have elements in $\mathcal{C}(V, \mathbf{g})$. If we call $J \subset I$ the subset of I such that $\{\varphi_s\}_{s \in J} \subset \mathcal{C}(V, \mathbf{g})$ then we get the following relationship between J and I if $J \supseteq [0, \epsilon)$ for $\epsilon > 0$.

Proposition 3.1 *If $\{\varphi_s\}_{s \in I} \cap \mathcal{C}(V, \mathbf{g}) \supset \{\varphi_s\}_{s \in \bar{I}}$ with $\bar{I} = [0, \epsilon)$ then $\{\varphi_s\}_{s \in \mathbb{R}^+ \cap I} \subset \mathcal{C}(V, \mathbf{g})$.*

Proof : This is a straightforward consequence of the fact that every real number in I can be written as a finite sum of numbers in \bar{I} . \square

Therefore, we deduce that under the above conditions $\{\varphi_s\}_{s \in \mathbb{R}^+ \cap I}$ is itself a *local submonoid of causal symmetries* (we will sometimes drop the tag “local” for the sake of brevity). We will work with these structures upon which we will try to carry out the generalization program mentioned before.

Proposition 3.2 *Let $\{\varphi_s\}_{s \in \mathbb{R}^+ \cap I}$ be a local submonoid of causal symmetries and suppose that the diffeomorphism φ_{s_0} for $|s_0| \in \mathbb{R}^+ \cap I$ is a conformal transformation. Then $\{\varphi_s\}_{s \in I}$ is a local one-parameter group of conformal transformations.*

Proof : For any null vector \vec{k} we have according to theorem A.4 that $\varphi'_{|s_0|}\vec{k}$ must be also null since $\varphi_{|s_0|}$ is conformal. If we write this conformal transformation as $\varphi_{|s_0|} = \varphi_{s_1} \circ \varphi_{s_2}$, $s_1, s_2 > 0$, we immediately get, using proposition A.9, that $\varphi'_{s_2}\vec{k}$ must be null from what we deduce that $\varphi'_s\vec{k}$ must be null if $0 \leq s \leq s_0$. Since this has been done for an arbitrary null vector, we conclude using again theorem A.4 that φ_s are conformal transformations for $0 \leq s \leq s_0$ and hence $\forall s \in I$ in view of the group properties of conformal transformations. \square

This proposition tells us that one-parameter submonoids of causal symmetries either consist exclusively of conformal transformations (and they can be extended to one-parameter groups recovering the classical theory) or they contain just the identity as the unique conformal transformation. We will henceforth study one-parameter submonoids with the identity as the only conformal transformation.

Corollary 3.1 *There are no realizations of S^1 as a one-parameter submonoid of proper causal symmetries.*

Proof : We can prove this by noting that if S^1 were realized as a one-parameter submonoid of proper causal symmetries then the parameter s labelling each diffeomorphism could be chosen running in the interval $[0, 2\pi]$ with $Id = \varphi_0 = \varphi_{2\pi}$. Proposition 3.2 would imply then that φ_s would be a conformal transformation $\forall s \in [0, 2\pi]$. \square

We can put this result another way by saying that proper causal symmetries cannot have cyclic orbits.

3.1. Canonical null directions of submonoids of causal symmetries

We will extend now the study of the set of canonical null directions in the previous section to the case of local continuous one-parameter submonoids of causal symmetries $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+}$. To start with, we prove one of our fundamental results.

Theorem 3.1 *If $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+} \subset \mathcal{C}(V, \mathbf{g})$ and \mathbf{g} is analytic we have the property $\mu(\varphi_{s_0}^* \mathbf{g}|_x) = \mu(\varphi_{s_1}^* \mathbf{g}|_x) \forall s_0, s_1 > 0$.*

Proof : If $s_0 > 0$, proposition 2.1 implies that $\forall s_1 \in [0, s_0]$, $\mu(\varphi_{s_0})|_x = \mu(\varphi_{s_0-s_1} \circ \varphi_{s_1})|_x \subseteq \mu(\varphi_{s_1})|_x$ and hence $\mu(\varphi_{s_0}^* \mathbf{g}|_x) \subseteq \mu(\varphi_{s_1}^* \mathbf{g}|_x)$. Now if $\vec{k} \in \mu(\varphi_{s_1}^* \mathbf{g}|_x)$ and we define the function $f_{\vec{k}}(s) \equiv \varphi_s^* \mathbf{g}|_x(\vec{k}, \vec{k})$ with \vec{k} fixed, we deduce that $f_{\vec{k}}(s) = 0$ for every $s \in I$ since this is an analytic function vanishing on the interval $[0, s_1]$ (due to the previous stated inclusion). This means that $\vec{k} \in \mu(\varphi_{s_0}^* \mathbf{g}|_x)$ from what the inclusion $\mu(\varphi_{s_1}^* \mathbf{g}|_x) \subseteq \mu(\varphi_{s_0}^* \mathbf{g}|_x)$ follows. \square

Remarks.

- (i) The proof of this proposition implies in fact that, for an analytic metric \mathbf{g} , $0 = \varphi_s^* \mathbf{g}|_x(\vec{k}, \vec{k}) \forall s \in I$ if $\vec{k} \in \mu(\varphi_{s_0}^* \mathbf{g}|_x)$ for a certain $s_0 > 0$.
- (ii) Observe that the elements of $\mu(\varphi_s^* \mathbf{g})$, $s > 0$ are the null vectors \vec{k} which remain null under the action of $\{\varphi_s\}_{s \in I}$, that is, such that $\varphi'_s \vec{k}$ is null for all $s \in I$. Thus, this can be taken as definition of *canonical null directions* of the submonoid.

We have just proven that the principal null directions of $\varphi_s^* \mathbf{g}$ do not depend on the value of $s > 0$ provided \mathbf{g} is analytic. Nonetheless the algebraic type of $\varphi_s^* \mathbf{g}$, and the number of such directions, might change from point to point and so we can define the sets $\mathcal{N}_{\varphi_s}^m$, \mathcal{U}_{φ_s} and \mathcal{N}_{φ_s} which split the manifold in sets with a constant number of them for the tensor $\varphi_s^* \mathbf{g}$. The previous proposition can be rewritten in terms of these sets as $\mathcal{N}_{\varphi_{s_1}}^m = \mathcal{N}_{\varphi_{s_2}}^m \equiv \mathcal{N}^m$, $\mathcal{N}_{\varphi_{s_1}} = \mathcal{N}_{\varphi_{s_2}} \equiv \mathcal{N}$, $\mathcal{U}_{\varphi_{s_1}} = \mathcal{U}_{\varphi_{s_2}} \equiv \mathcal{U}$, $\forall s_1, s_2 \in I \cap \mathbb{R}^+$ but we can even say more about these sets.

Proposition 3.3 *If \mathbf{g} is analytic then for every one-parameter submonoid of causal symmetries $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+}$ we have that*

$$\varphi_s(\mathcal{U}) = \mathcal{U}, \quad \varphi_s(\mathcal{N}^m) = \mathcal{N}^m, \quad \varphi_s(\mathcal{N}) = \mathcal{N} \quad \forall s \in I.$$

Proof : To prove this proposition we must show that the set of canonical null directions is preserved under the flow of $\{\varphi_s\}_{s \in I}$, that is to say, they are the same for the tensors $(\varphi_{s_0}^* \mathbf{g})|_x$ and $(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)}$ (notice that in view of theorem 3.1 the precise s_0 has no relevance here as long as it is kept positive). Thus, we are going to establish a bijection between the set of null eigenvectors of $(\varphi_{s_0}^* \mathbf{g})|_x$ and those of $(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)}$. Let us start by writing down the equation

$$(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)} (\varphi'_s \vec{\mathbf{u}}_1, \varphi'_s \vec{\mathbf{u}}_2) = (\varphi_{s_0+s}^* \mathbf{g})|_x (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2), \quad \forall \vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2 \in T(V), \quad s \in I, \quad (3.1)$$

and choosing any null vector $\vec{\mathbf{k}} \in \mu(\varphi_{s_0}^* \mathbf{g})|_x$. Theorem 3.1 implies that $\vec{\mathbf{k}} \in \mu(\varphi_{s_0+s}^* \mathbf{g})|_x$ for $s > 0$ and hence use of (3.1) tells us that $\varphi'_s \vec{\mathbf{k}} \in \mu(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)}$ which means that φ'_s sets an injection between $\mu(\varphi_{s_0}^* \mathbf{g})|_x$ and $\mu(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)}$. To show that φ'_s is also onto, we pick any $\vec{\mathbf{u}} \in \mu(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)}$ and look again at the proof of theorem 3.1 deducing that the vector $\vec{\mathbf{k}}$ such that $\varphi'_s \vec{\mathbf{k}} = \vec{\mathbf{u}}$ must also be null $\forall s \in I \cap \mathbb{R}^+$ so that inserting this expression of $\vec{\mathbf{u}}$ into equation (3.1) we conclude that $0 = (\varphi_{s_0+s}^* \mathbf{g})|_x (\vec{\mathbf{k}}, \vec{\mathbf{k}})$ which means according to theorem 3.1 that $\vec{\mathbf{k}} \in \mu(\varphi_{s_0}^* \mathbf{g})|_x$. Therefore the push-forward φ'_s sets the sought bijection between $\mu(\varphi_{s_0}^* \mathbf{g})|_x$ and $\mu(\varphi_{s_0}^* \mathbf{g})|_{\varphi_s(x)}$ which leads us to the required invariance properties of \mathcal{U} and \mathcal{N}^m for $s > 0$ and by extension $\forall s \in I$ due to the group property of $\{\varphi_s\}_{s \in I}$ (for example, $\varphi_s(\mathcal{U}) = \mathcal{U} \Rightarrow \mathcal{U} = \varphi_{-s}(\mathcal{U})$.) \square

Remarks.

- (i) $\forall x \notin X$ (see (2.3)), its orbit $\mathcal{O}_x = \{y \in V : \varphi_s(x) = y\}$ is a subset of one of the invariant sets \mathcal{N}^m or \mathcal{U} .
- (ii) Theorem 3.1 and proposition 3.3 allow us to speak about the canonical null directions of a one-parameter submonoid of causal symmetries independently of the parameter s or of the region of the manifold (once we have chosen one of the sets \mathcal{U} or \mathcal{N}^m). We will work in the sequel with manifolds with just a single component \mathcal{U} or \mathcal{N}^m and the metric tensor will be assumed analytic there unless otherwise stated. By reasons which shall be clear later, we will denote the set of canonical null directions at a point as $\mu_{\vec{\xi}}|_x$ being $\vec{\xi}$ the generating vector field of $\{\varphi_s\}_{s \in I}$. As always we may give the global definition of this as $\mu_{\vec{\xi}} = \bigcup_{x \in V} \mu_{\vec{\xi}}|_x$ which is a subset of the bundle $T(V)$ and we will use the same symbol if there is need of considering sections over $\mu_{\vec{\xi}}$. Theorem 3.1 tells us that

$$\varphi'_s(\mu_{\vec{\xi}}) = \mu_{\vec{\xi}}, \quad \forall s \in I. \quad (3.2)$$

As happened with the set of principal directions, $\mu_{\vec{\xi}}|_x$ is not a vector space but we can pick up a maximum number of linearly independent elements denoted by $\chi(\mu_{\vec{\xi}}|_x)$ which is simply $\dim \text{Span}(\mu_{\vec{\xi}}|_x)$.

- (iii) The maximum number of linearly independent canonical null directions can be used as a first mean to classify these symmetries as they were used to classify causal relations in PII. The case with $\chi(\mu_{\vec{\xi}}|_x) = n, \forall x \in V$ is given by the conformal transformations of the manifold V whereas the case with $\chi(\mu_{\vec{\xi}}|_x) = m < n, \forall x \in V$ shall be called partly conformal case and its elements $\frac{m}{n}$ -partly conformal symmetries as they do not preserve the full null cone. The most simple case of nontrivial partly conformal symmetries is the one with $m = n - 1$ (we will give in proposition 4.2 a very simple differential condition for the generating vectors of these symmetries). The complexity increases as the number m decreases. A one-parameter submonoid of causal symmetries with a single linearly independent canonical null direction is called *degenerate*, and *nondegenerate* otherwise.

The results of proposition 2.5 can also be extended to the case of one-parameter submonoids of causal symmetries as follows:

Proposition 3.4 *If $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+} \subset \mathcal{C}(V, \mathbf{g})$, then $\varphi'_s[\mu(\varphi_s^* \mathbf{T}|_x)] = \mu(\mathbf{T}|_{\varphi_s(x)}) \cap \mu_{\vec{\xi}}|_{\varphi_s(x)}$ $\forall \mathbf{T} \in \mathcal{DP}_r^+(V)$.*

Proof : The inclusion $\varphi'_s[\mu(\varphi_s^* \mathbf{T}|_x)] \subseteq \mu_{\vec{\xi}}|_{\varphi_s(x)} \cap \mu(\mathbf{T}|_{\varphi_s(x)})$ comes from proposition 2.5 (actually it is also true for non-analytic metrics). If $\vec{\mathbf{k}}$ is any null vector in $\mu_{\vec{\xi}}|_{\varphi_s(x)} \cap \mu(\mathbf{T}|_{\varphi_s(x)})$ we know that $\varphi'_{-s}\vec{\mathbf{k}} \in \mu_{\vec{\xi}}|_x \forall s \in I$ and we have $0 = \mathbf{T}|_{\varphi_s(x)}(\vec{\mathbf{k}}, \dots, \vec{\mathbf{k}}) = (\varphi_s^* \mathbf{T})|_x(\varphi'_{-s}\vec{\mathbf{k}}, \dots, \varphi'_{-s}\vec{\mathbf{k}})$ which means for $s > 0$ that $\varphi'_{-s}\vec{\mathbf{k}} \in \mu(\varphi_s^* \mathbf{T}|_x)$ from what we conclude that $\varphi'_{-s}(\mu_{\vec{\xi}}|_{\varphi_s(x)} \cap \mu(\mathbf{T}|_{\varphi_s(x)})) \subseteq \mu(\varphi_s^* \mathbf{T}|_x)$ leading to the other inclusion. \square

As an application, take any causal tensor \mathbf{T} such that $\mu(\mathbf{T}|_x) \cap \mu_{\vec{\xi}}|_x = \mu_{\vec{\xi}}|_x, \forall x \in V$. This proposition implies then that $\mu(\varphi_s^* \mathbf{T}|_x) = \mu_{\vec{\xi}}|_x, \forall s \in I \cap \mathbb{R}^+, \forall x \in V$. The opposite case occurs when $\mu_{\vec{\xi}} = \emptyset$ which entails $\mu(\varphi_s^* \mathbf{T}|_x) = \emptyset, \forall x \in V, \forall s \in I \cap \mathbb{R}^+$ and for all $\mathbf{T} \in \mathcal{DP}_r^+$ (this case is true regardless of the analytic properties of the background metric). Another important outcome of this is the following.

Proposition 3.5 *If $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+} \subset \mathcal{C}(V, \mathbf{g})$ then for every $\mathbf{T} \in \mathcal{DP}_2^+(V)$ such that $\mu_{\vec{\xi}}|_x \subseteq \mu(\mathbf{T}|_x)$ and $\mathbf{T}_1 \times_1 \mathbf{T} \neq 0$ we have that $\sigma(\varphi_s^* \mathbf{T}|_x) = \mu(\varphi_s^* \mathbf{T}|_x) = \mu_{\vec{\xi}}|_x, \forall s > 0$.*

Proof : This follows from the already proven inclusions $\mu(\varphi_s^* \mathbf{T}|_x) \subseteq \sigma(\varphi_s^* \mathbf{T}|_x) \subseteq \mu(\varphi_s^* \mathbf{g}|_x) = \mu_{\vec{\xi}}|_x, s > 0$ (see proposition 2.5) and the previous proposition. \square

Remark. Note that the property $\sigma(\varphi_s^* \mathbf{T}|_x) = \mu(\varphi_s^* \mathbf{T}|_x)$ holds with no assumptions on $\mu_{\vec{\xi}}|_x$ if \mathbf{T} has Lorentzian signature (see proposition 2.6).

The invariance property (3.2) can be recast in a more handleable way.

Lemma 3.1 *Let $\vec{\mathbf{k}}^1, \dots, \vec{\mathbf{k}}^m \in \mu_{\vec{\xi}}$ be linearly independent with $m \equiv \chi(\mu_{\vec{\xi}})$. Then $\varphi'_s \vec{\mathbf{k}}^j|_{\varphi_s(x)} \in \text{Span}\{\vec{\mathbf{k}}^1|_{\varphi_s(x)}, \dots, \vec{\mathbf{k}}^m|_{\varphi_s(x)}\}$ and $\varphi_s^* \mathbf{k}^j|_x \in \text{Span}\{\mathbf{k}^1|_x, \dots, \mathbf{k}^m|_x\} \forall j = 1, \dots, m, s \in \mathbb{R}^+ \cap I$.*

Proof : The invariance law (3.2) says that $\varphi'_s(\mu_{\vec{\xi}}|_x) = \mu_{\vec{\xi}}|_{\varphi_s(x)}$ so for every $\vec{\mathbf{k}}|_x \in \mu_{\vec{\xi}}|_x$ we have that $\varphi'_s \vec{\mathbf{k}}|_{\varphi_s(x)} \in \mu_{\vec{\xi}}|_{\varphi_s(x)}$ which means that $\varphi'_s \vec{\mathbf{k}}|_{\varphi_s(x)} \in \text{Span}\{\vec{\mathbf{k}}^1|_{\varphi_s(x)}, \dots, \vec{\mathbf{k}}^m|_{\varphi_s(x)}\}$ since by assumption these vectors are a basis of the subspace generated by $\mu_{\vec{\xi}}|_{\varphi_s(x)}$. Now the property $\mu_{\vec{\xi}}|_x = \varphi'_{-s}(\mu_{\vec{\xi}}|_{\varphi_s(x)})$ leads us to a similar statement about $\varphi'_{-s} \vec{\mathbf{k}}|_x$ if $\vec{\mathbf{k}}|_{\varphi_s(x)} \in \mu_{\vec{\xi}}|_{\varphi_s(x)}$ and hence $\varphi'_{-s} \vec{\mathbf{k}}|_x \in \text{Span}\{\vec{\mathbf{k}}^1|_x, \dots, \vec{\mathbf{k}}^m|_x\}$. This last result is necessary to prove the property involving

the differential forms because if we pull-back the defining relation of each $\mathbf{k}|_x$ in terms of $\vec{\mathbf{k}}|_x$ we get:

$$k_a = g_{ab}k^b \Rightarrow (\varphi_s^*k)_a = (\varphi_s^*g)_{ab}(\varphi'_{-s}k)^b, \quad (3.3)$$

where an appropriate generalization of (2.1) has been used. The sought result follows from the fact that $\varphi'_{-s}\vec{\mathbf{k}}|_x$ ($s > 0$) is a null eigenvector of $\varphi_s^*\mathbf{g}|_x$ which belongs to $\text{Span}\{\vec{\mathbf{k}}^1|_x, \dots, \vec{\mathbf{k}}^m|_x\}$. \square

It is clear that these and other forthcoming properties can be generalized to every $s \in I$ but for our present purposes it is enough to stick to positive values of s . The relevance of these invariance properties shall be addressed later on. To that end we rewrite them in a more geometrical and compact way.

Proposition 3.6 *If $\{\vec{\mathbf{k}}^1, \dots, \vec{\mathbf{k}}^m\}$ and $\{\mathbf{k}^1, \dots, \mathbf{k}^m\}$, are the set of vectors and forms introduced in the previous lemma then the m -vector $\vec{\Omega} = \vec{\mathbf{k}}^1 \wedge \dots \wedge \vec{\mathbf{k}}^m$ and the m -form $\Omega = \mathbf{k}^1 \wedge \dots \wedge \mathbf{k}^m$ behave under the flow of $\{\varphi_s\}$ as:*

$$\varphi'_s \vec{\Omega}|_{\varphi_s(x)} = \frac{\lambda_s^m(\varphi_s(x))}{\sigma_s(\varphi_s(x))} \vec{\Omega}|_{\varphi_s(x)}, \quad \varphi_s^* \Omega|_x = \sigma_s(x) \Omega|_x, \quad s \in I \cap \mathbb{R}^+, \quad (3.4)$$

where $\lambda_s(x)$ is the common eigenvalue of the elements of $\mu_{\vec{\xi}}$.

Proof : Under the hypotheses of lemma 3.1 we deduce by means of $\varphi'_s(\vec{\mathbf{k}}^1 \wedge \dots \wedge \vec{\mathbf{k}}^m)|_{\varphi_s(x)} = \varphi'_s \vec{\mathbf{k}}^1|_{\varphi_s(x)} \wedge \dots \wedge \varphi'_s \vec{\mathbf{k}}^m|_{\varphi_s(x)}$ and $\varphi_s^*(\mathbf{k}^1 \wedge \dots \wedge \mathbf{k}^m)|_x = \varphi_s^* \mathbf{k}^1|_x \wedge \dots \wedge \varphi_s^* \mathbf{k}^m|_x$ that $\varphi'_s \vec{\Omega}|_{\varphi_s(x)} \propto \vec{\Omega}|_{\varphi_s(x)}$ and $\varphi_s^* \Omega|_x \propto \Omega|_x$. We can freely set one of the proportionality factors appearing in these equations so we put $\varphi_s^* \Omega|_x = \sigma_s(x) \Omega|_x$. The other proportionality factor called β_s is found by pulling back the equation $\Omega_{a_1 \dots a_m}(x) = g_{a_1 b_1}(x) \dots g_{a_m b_m}(x) \Omega^{b_1 \dots b_m}(x)$ which leads us to

$$\varphi'_s \Omega_{a_1 \dots a_m}(x) = (\varphi_s^* g)_{a_1 b_1}(x) \dots (\varphi_s^* g)_{a_m b_m}(x) (\varphi'_{-s} \Omega)^{b_1 \dots b_m}(x). \quad (3.5)$$

The expression of $\varphi'_{-s} \vec{\Omega}|_x$ can be worked out from $\varphi'_s \vec{\Omega}|_{\varphi_s(x)} = \beta_s(\varphi_s(x)) \vec{\Omega}|_{\varphi_s(x)}$ and it turns out to be $\varphi'_{-s} \vec{\Omega}|_x = \beta_s^{-1}(x) \vec{\Omega}|_x$. On the other hand, since the null vectors $\vec{\mathbf{k}}^j|_x$ forming the m -form are eigenvectors of $\varphi_s^* \mathbf{g}|_x$ with the same eigenvalue λ_s for $s > 0$ we have the property $(\varphi_s^* g)_{ab}(x) k^j{}^b(x) = \lambda_s(x) k^j{}_a(x)$. The insertion of both results in (3.5) leads to (3.4) at once. \square

In most of our calculations we will use the equation involving Ω rather than its vector counterpart. In the particular cases of $\mu_{\vec{\xi}}$ consisting of one or two linearly independent null vectors lemma 3.1 can be written as $\varphi'_s \vec{\mathbf{k}}|_{\varphi_s(x)} \propto \vec{\mathbf{k}}|_{\varphi_s(x)}$ with $\vec{\mathbf{k}}$ any element of $\mu_{\vec{\xi}}$.

An important rank-2 tensor related with the m -form Ω introduced in proposition 3.6 is its Lorentz tensor $T\{\Omega\} = \mathbf{S}$ which, as explained in appendix A, is an element of $\mathcal{DP}_2^+(V)$ which can be normalized as $S_{ac} S_b^c = g_{ab}$ by choosing Ω such that $\Omega \cdot \Omega = \Omega_{a_1 \dots a_m} \Omega^{a_1 \dots a_m} = (-1)^{m-1} 2m!$ (unless $\Omega \cdot \Omega = 0$ in which case $S_{ac} S_b^c = 0 \Leftrightarrow \mathbf{S} = f \mathbf{k} \otimes \mathbf{k}$, \mathbf{k} null; in our setting this only happens in the degenerate case). The only possible eigenvalues for the normalized tensor \mathbf{S} arising in the nondegenerate case are +1 and -1.

We can now write down equations for $\varphi_s^* \mathbf{g}$ when we have a one-parameter submonoid of causal symmetries. The very first thing we know is that $\varphi_s^* \mathbf{g} \in \mathcal{DP}_2^+(V)$ so we can apply theorem A.1 to this tensor and write it as

$$\varphi_s^* \mathbf{g} = \alpha_s \mathbf{g} + \mathbf{U}_s, \quad \alpha_s > 0,$$

where $\mathbf{U}_s \in \mathcal{DP}_2^+$ gathers the terms of the decomposition which are not proportional to the metric tensor. The term proportional to the metric always appears if s is small enough due to the continuity in the parameter s (actually it can be shown by algebraic arguments that $\alpha_s \neq 0$, $\forall s \in \mathbb{R}^+ \cap I$) whence $\mu(\varphi_s^* \mathbf{g}) = \mu(\mathbf{U}_s) = \mu_{\vec{\xi}}$, $\forall s \in I \cap \mathbb{R}^+$. In the case of more than one linearly independent canonical null direction, a new application of theorem A.1 to \mathbf{U}_s tells us that \mathbf{S} will always be the first term of the decomposition so it would be useful to know how it behaves under the pull-back of φ_s . This calculation is easily done using equation (A.1) with $\Sigma = \Omega$ which yields the formula for the nondegenerate case with $\chi(\mu_{\vec{\xi}}) = m > 1$:

$$(\varphi_s^* S)_{ab}(x) = \frac{(-1)^{m-1}}{(m-1)!} \left[(\varphi_s^* \Omega)_{aa_2 \dots a_m}(x) (\varphi_s^* \mathbf{g})_{ba_1}(x) (\varphi'_{-s} \Omega)^{a_1 a_2 \dots a_m}(x) - \frac{1}{2m} (\varphi_s^* \mathbf{g})_{ab}(x) (\varphi_s^* \Omega)_{a_1 \dots a_m}(x) (\varphi'_{-s} \Omega)^{a_1 \dots a_m}(x) \right],$$

where everything is known. Moreover is quite simple to realize that the normalization imposed on Ω in this case allows us to rewrite its invariance law as $\varphi'_{-s} \vec{\Omega}|_x = \vec{\Omega}|_x / \sigma_s(x)$ by just pulling back the normalization condition. Plugging $\varphi_s^* \Omega|_x$ and $\varphi'_{-s} \vec{\Omega}|_x$ into the previous equation we get:

$$(\varphi_s^* S)_{ab} = \frac{(-1)^{m-1}}{(m-1)!} \left[\Omega_{aa_2 \dots a_m} (\varphi_s^* \mathbf{g})_{ba_1} \Omega^{a_1 a_2 \dots a_m} - \frac{1}{2m} (\varphi_s^* \mathbf{g})_{ab} \Omega \cdot \Omega \right].$$

Replacing $\varphi_s^* \mathbf{g}$ by $\alpha_s \mathbf{g} + \mathbf{U}_s$ and rearranging terms we obtain the final formula

$$(\varphi_s^* S)_{ab} = \alpha_s S_{ab} + (U_s)_{bc} S_a^c, \quad \chi(\mu_{\vec{\xi}}) > 1, \quad (3.6)$$

from where the property $(U_s)_{bc} S_a^c = (U_s)_{ac} S_b^c$ follows due to the symmetry of $(\varphi_s^* S)_{ab}$.

The degenerate case is even simpler as $\mathbf{S} = \mathbf{k} \otimes \mathbf{k}$ (we set $f = 1$) and we know that $\varphi_s^* \mathbf{k}|_x = \gamma_s(x) \mathbf{k}|_x$ for some function $\gamma_s(x)$. Whence the transformation law for $\mathbf{S} = \mathbf{k} \otimes \mathbf{k}$ in the degenerate case is simply

$$\varphi_s^* \mathbf{S}|_x = \gamma_s^2(x) \mathbf{S}|_x, \quad \chi(\mu_{\vec{\xi}}) = 1. \quad (3.7)$$

If there are no canonical null directions then things are far more involved because we cannot prove by the same means the existence of a timelike future-directed vector field invariant under the submonoid $\{\varphi_s\}_{s \in I}$ of causal symmetries. Even if such a field $\vec{\mathbf{u}}$ did exist, the invariance property $\varphi'_s(\vec{\mathbf{u}}|_x) \propto \vec{\mathbf{u}}|_{\varphi_s(x)}$, $s > 0$ would not imply that $\vec{\mathbf{u}}|_x$ is an eigenvector of $(\varphi_s^* \mathbf{g})|_x$ (see Example 1 below) as happened in the cases with $\chi(\mu_{\vec{\xi}}) = 1, 2$. Hence, in this last case we will have to deal with two different situations according to whether $\vec{\mathbf{u}}|_x$ is or not an eigenvector of $(\varphi_s^* \mathbf{g})|_x$.

Example 1. Let $(V, \mathbf{g}) = (\mathbb{L}, \boldsymbol{\eta})$ be flat Minkowski spacetime and consider any linear automorphism $\hat{\mathbf{T}}$ of \mathbb{L} seen as a vector space. Then $\hat{\mathbf{T}}$ defines in an obvious way a transformation $\varphi \in \text{Diff}(\mathbb{L})$ which, if the time orientation is preserved, will be in $\mathcal{C}(\mathbb{L}, \boldsymbol{\eta})$ whenever $(\varphi^* \boldsymbol{\eta})_{ab} = \hat{T}_a^p \eta_{pq} \hat{T}_b^q \in \mathcal{DP}_2^+(\mathbb{L})$. Clearly it is possible to define one-parameter monoids of this type of transformations such that $\{\varphi_s\}_{s>0} \in \mathcal{C}(\mathbb{L}, \boldsymbol{\eta})$ and having the properties described in this section. An additional property holding in this case is the existence of an invariant causal direction for each value of s (see theorem B.2 of appendix B), so if we denote by $\vec{\mathbf{u}}_s$ such direction we get the property $\varphi'_s \vec{\mathbf{u}}_s = \hat{\mathbf{T}}_s(\vec{\mathbf{u}}_s) \propto \vec{\mathbf{u}}_s$. Nevertheless, in the case that $\{\varphi_s\}_{s>0}$ has no canonical null directions, the (necessarily) timelike vector $\vec{\mathbf{u}}_s$ will not be in general an eigenvector of $\varphi_s^* \boldsymbol{\eta}$ unless $(u_s)_a (\hat{T}_s)^a_b \propto (u_s)_b$.

In the next section we will finally derive the differential conditions we are seeking. They are the infinitesimal counterpart of the equations of this section and we shall discuss some of their properties.

4. Infinitesimal theory

Definition 4.1 A smooth vector field $\vec{\xi}$ defined on an entire Lorentzian manifold is said to be causal preserving if the one-parameter group $\{\varphi_s\}_{s \in I}$ generated by $\vec{\xi}$ is such that $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+} \in \mathcal{C}(V, \mathbf{g})$. A casual-preserving vector field is called (non-) degenerate if the corresponding $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+}$ is (non-) degenerate.

Remark. This definition is rather general in the sense that $\{\varphi_s\}_{s \in \mathbb{R}^+ \cap I}$ may give rise to different components \mathcal{U} and \mathcal{N} of V with $\chi(\mu_{\vec{\xi}}|_x)$ changing from one set to another. Proposition 3.3 allows us to study only cases in which V consists of just one of these sets and so all the causal-preserving vector fields (CPVF from now on) of this paper will be understood in this way.

As stated in the previous section, our aim is to find differential conditions for the causal symmetries analogous to those of other known symmetries. We need a lemma.

Lemma 4.1 Let $\{\mathbf{T}_s\}_{s \in I}$ be a one-parameter family of rank- r covariant (contravariant) tensors differentiable in the parameter s such that $\mathbf{T}_{s_0} = \mathbf{0}$ and assume further that $\mathbf{T}_s \in \mathcal{DP}_r^+(V)$ for $s \in [s_0, s_0 + \epsilon)$. Then $d\mathbf{T}_s/ds|_{s=s_0} \equiv \dot{\mathbf{T}}_{s_0} \in \mathcal{DP}_r^+(V)$ (or its contravariant counterpart).

Proof : Define the family of functions $f_{\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r}(s) = \mathbf{T}_s(\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r)$ for each r -tuple of causal future-directed vectors $\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r$. All these functions vanish at $s = s_0$ and are non-negative if s lies in $[s_0, s_0 + \epsilon)$ due to the hypotheses of the lemma so they are nondecreasing at $s = s_0$ as functions of the parameter s which entails $d\mathbf{T}_s/ds|_{s=s_0}(\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r) \equiv f'_{\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r}(s_0) = \dot{\mathbf{T}}_{s_0}(\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r) \geq 0$ for every set of causal future-directed vectors $\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r$. \square

The next proposition characterizes $\mu_{\vec{\xi}}$ as the set of null vectors onto which $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ vanishes so that this set depends only on the CPVF $\vec{\xi}$.

Proposition 4.1 $\vec{\mathbf{k}} \in \mu_{\vec{\xi}}|_x$ if and only if $(\mathcal{L}_{\vec{\xi}} \mathbf{g})|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$.

Proof : The “only if” implication comes easily from the property $\varphi_s^* \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$ holding for every $\vec{\mathbf{k}} \in \mu_{\vec{\xi}}$. To prove the “if” assertion we proceed by contradiction: let us assume that a null vector $\vec{\mathbf{k}}|_x$ exists such that $\varphi_s^* \vec{\mathbf{k}}|_{\varphi_s(x)}$ is timelike for $s \in \mathbb{R}^+ \cap I$ and $\mathcal{L}_{\vec{\xi}} \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$. Lemma 2.5 in PI or a direct calculation shows that $\varphi_s^* \vec{\mathbf{k}}|_{\varphi_s(x)} = c_s \vec{\mathbf{k}}|_{\varphi_s(x)} + \vec{\mathbf{n}}_s|_{\varphi_s(x)}$ for some $c_s > 0$ and $\vec{\mathbf{n}}_s|_{\varphi_s(x)}$ null and future directed. Therefore $\vec{\mathbf{n}}_s|_x = \varphi_s^* \vec{\mathbf{k}}|_x - c_s \vec{\mathbf{k}}|_x$ is a one-parameter family of causal vectors which comply with the hypotheses of lemma 4.1 for $s_0 = 0$ so $\dot{\vec{\mathbf{n}}}_0|_x = -\mathcal{L}_{\vec{\xi}} \vec{\mathbf{k}}|_x - \dot{c}_0 \vec{\mathbf{k}}|_x \in \Theta_x^+$. On the other hand if we apply the Lie derivative to $k^a g_{ab} k^b$ and use $0 = (\mathcal{L}_{\vec{\xi}} \mathbf{g})_{ab} k^a k^b$ we get $k_a \mathcal{L}_{\vec{\xi}} k^a = 0$ so $\dot{\vec{\mathbf{n}}}_0|_x$ is causal and orthogonal to $\vec{\mathbf{k}}|_x$ which is only possible if both vectors are null and proportional to each other, that is to say, $\mathcal{L}_{\vec{\xi}} \vec{\mathbf{k}}|_x \propto \vec{\mathbf{k}}|_x$ which integrated yields $\varphi_s^* \vec{\mathbf{k}}|_{\varphi_s(x)} \propto \vec{\mathbf{k}}|_{\varphi_s(x)}$ against the assumption of $\varphi_s^* \vec{\mathbf{k}}$ being timelike. \square

These two results and the calculations of the last part of the previous section is all what is needed to establish one of the main results of this paper.

Theorem 4.1 Let $\{\varphi_s\}_{s \in I \cap \mathbb{R}^+}$ be a one-parameter submonoid of causal symmetries generated by the vector field $\vec{\xi}$. Then $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ and $\mathcal{L}_{\vec{\xi}} \mathbf{S}$ adopt the expressions written below, according to the number of linearly independent elements of $\mu_{\vec{\xi}}$:

(i) if $\chi(\mu_{\vec{\xi}}) > 1$ then:

$$\mathcal{L}_{\vec{\xi}} g_{ab} = \alpha g_{ab} + \beta S_{ab} + Q_{ab}, \quad (4.1)$$

$$\mathcal{L}_{\vec{\xi}} S_{ab} = \beta g_{ab} + \alpha S_{ab} + Q_{ac} S^c_b, \quad (4.2)$$

(ii) if $\chi(\mu_{\vec{\xi}}) = 1$ (degenerate CPVFs) then:

$$\mathcal{L}_{\vec{\xi}} g_{ab} = \alpha g_{ab} + \beta k_a k_b + Q_{ab}, \quad \mathcal{L}_{\vec{\xi}} S_{ab} = 2\gamma S_{ab} \Leftrightarrow \mathcal{L}_{\vec{\xi}} k_a = \gamma k_a, \quad (4.3)$$

with $S_{ab} = k_a k_b$.

(iii) and, if there are no canonical null directions:

$$\mathcal{L}_{\vec{\xi}} g_{ab} = \alpha g_{ab} + U_{ab}. \quad (4.4)$$

In all cases $\mathbf{U}, \mathbf{Q} \in \mathcal{DP}_2^+(V)$, $\mu(\mathbf{Q}) \supset \mu_{\vec{\xi}}$ and $\beta > 0$.

Proof : The family $\{\mathbf{U}_s\}_{s \in I}$ introduced in previous section behaves in the way of lemma 4.1 with $s_0 = 0$ so the formula for $\varphi_s^* \mathbf{g}$ in terms of \mathbf{U}_s yields the equation:

$$\left. \frac{d\varphi_s^* \mathbf{g}}{ds} \right|_{s=0} = \mathcal{L}_{\vec{\xi}} \mathbf{g} = \alpha \mathbf{g} + \mathbf{U}, \quad \alpha = \left. \frac{d\alpha_s}{ds} \right|_{s=0}, \quad \mathbf{U} = \left. \frac{d\mathbf{U}_s}{ds} \right|_{s=0}. \quad (4.5)$$

In the case of $\chi(\mu_{\vec{\xi}}) > 1$, this equation must be supplemented with the derivative with respect to s of (3.6):

$$\left. \frac{d\varphi_s^* S_{ab}}{ds} \right|_{s=0} = \mathcal{L}_{\vec{\xi}} S_{ab} = \alpha S_{ab} + (U)_{ac} S^c_b, \quad (4.6)$$

where the tensor $\mathbf{U} \in \mathcal{DP}_2^+(V)$. Equation (4.5) and proposition 4.1 imply that $\mu(\mathbf{U}) = \mu_{\vec{\xi}}$ so if we apply theorem A.1 to \mathbf{U} we may write it as $\mathbf{U} = \beta \mathbf{S} + \mathbf{Q}$ where β must be strictly positive since it is the coefficient of the first term of the decomposition and \mathbf{Q} is a causal tensor collecting the rest of the terms, and thus $\mu_{\vec{\xi}} = \mu(\mathbf{U}) \subset \mu(\mathbf{Q})$. Inserting this new expression of \mathbf{U} into equations (4.5) and (4.6) yields (4.1) and (4.2). In the degenerate case \mathbf{U} has only a single null direction so that $\mathbf{S} \propto \mathbf{k} \otimes \mathbf{k}$. The second equation of (4.3) is the derivative of (3.7) at $s = 0$. Finally, in the case with no canonical null directions we are only left with equation (4.5) where now \mathbf{U} is a causal tensor with no principal directions. \square

Remarks.

- (i) As we explained in section 2 the set $\mathcal{C}(V, \mathbf{g})$ is invariant under conformal rescalings of the metric tensor \mathbf{g} . Therefore, if $\{\varphi_s\}_{s \in \mathbb{R}^+ \cap I}$ is a submonoid of causal symmetries with respect to $\mathcal{C}(V, \mathbf{g})$ it will also be so for any metric $\sigma \mathbf{g}$, $\sigma(x) > 0 \forall x \in V$ (with the same number of linearly independent canonical null directions as is obvious). As a result of this conformal invariance the function $\alpha(x)$ which appears in equations (4.1-4.4) is redefined by conformal rescaling of the metric $\mathbf{g} \rightarrow \sigma \mathbf{g}$ as $\alpha \rightarrow \sigma^{-1} \mathcal{L}_{\vec{\xi}} \sigma + \alpha$ whereas \mathbf{Q}, \mathbf{U} and \mathbf{S} are just rescaled by σ . Moreover, we can set $\sigma^{-1} \mathcal{L}_{\vec{\xi}} \sigma + \alpha$ to zero by choosing an appropriate initial condition for σ . This choice can only be made globally if the vector $\vec{\xi}$ is complete.

- (ii) The remainder \mathbf{Q} will depend on each CPVF although some general properties are common to every case. Given that $\mu_{\tilde{\xi}} \subset \mu(\mathbf{Q})$ if $\mu_{\tilde{\xi}} \neq \emptyset$, every element of $\mu_{\tilde{\xi}}$ is a null eigenvector of \mathbf{Q} which implies that

$$Q_a^b \Omega_{ba_2 \dots a_m} = \lambda \Omega_{aa_2 \dots a_m} \Rightarrow Q_{ab} S_c^b = Q_{cb} S_a^b,$$

where λ is the associated eigenvalue. In the case of $\mu_{\tilde{\xi}}$ being empty, it is still possible to get an equation similar to (4.1) as follows: since the tensor U_{ab} of equation (4.4) has no principal directions, the first term of its canonical decomposition βS_{ab} , $\beta > 0$, is now proportional to the Lorentz tensor S_{ab} associated to the timelike eigenvector field $\tilde{\mathbf{u}}$ of $\mathcal{L}_{\tilde{\xi}} \mathbf{g}$ (we also keep the notation Q_{ab} for the remainder appearing in this decomposition). Assuming the normalization $u_a u^a = 1$, this tensor reads $S_{ab} = 2u_a u_b - g_{ab}$. As we commented in example 1, invariance properties under the one-parameter submonoid $\{\varphi_s\}_{s \in \mathbb{R} + \cap I}$ are in general unknown in this case so, as far as we know, equation (4.2) will not be true in general. Nonetheless, in the particular case of having a timelike direction invariant under $\{\varphi_s\}_{s \in \mathbb{R} + \cap I}$, say $\tilde{\mathbf{v}}$, we can still use part of the calculations above. For instance the Lie derivative of the 1-form v_a is given by:

$$\mathcal{L}_{\tilde{\xi}} v_a = \mathcal{L}_{\tilde{\xi}} (g_{ap} v^p) = (\alpha g_{ap} + \beta S_{ap} + Q_{ap}) v^p + \gamma v_a, \quad (4.7)$$

where we have used $\mathcal{L}_{\tilde{\xi}} v^a = \gamma v^a$ for some function $\gamma(x)$. The case with $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}$ will be called *aligned* and has the same infinitesimal treatment as the others with $\chi(\mu_{\tilde{\xi}}) > 1$ because, as follows from the previous equation, $\mathcal{L}_{\tilde{\xi}} u_a \propto u_a$ and a very easy calculation leads then to (4.2) with the new Lorentz tensor S_{ab} . Therefore all the results arising from theorem 4.1 valid for the case with $\chi(\mu_{\tilde{\xi}}) > 1$ will also hold for the aligned case with the previous definition of \mathbf{S} in terms of $\tilde{\mathbf{u}} = \tilde{\mathbf{v}}$.

- (iii) Another important point is that $P_{ab} = (g_{ab} + S_{ab})/2$ is a projector onto the subspace $Span\{\mu(\mathbf{S})\} = \mu_{\tilde{\xi}}$ if \mathbf{S} is nondegenerate as it is explicitly shown in proposition A.3. This applies to case (i) of previous theorem, and also to the aligned case (with $\mu_{\tilde{\xi}} = \emptyset$) where the subspace is now $Span\{\tilde{\mathbf{u}}\}$. Since $Span\{\mu(\mathbf{S})\}$ (or $Span\{\tilde{\mathbf{u}}\}$) is an invariant subspace of Q_{ab} , this tensor must be invariant under the action of the projector which results in

$$Q_a^c (g_{cb} + S_{cb}) = \lambda (g_{ab} + S_{ab}).$$

With the aid of this result we can show explicitly the partly conformal character of these transformations by adding equations (4.1) and (4.2) to get:

$$\mathcal{L}_{\tilde{\xi}} P_{ab} = (\alpha + \beta) \frac{1}{2} (g_{ab} + S_{ab}) + Q_{ac} \frac{1}{2} (\delta_b^c + S_b^c) = (\alpha + \beta + \lambda) P_{ab}. \quad (4.8)$$

We see that these transformations can be regarded as conformal symmetries of the projector P_{ab} i.e. they are mapping conformally $Span\{\mu_{\tilde{\xi}}|_x\}$ ($Span\{\tilde{\mathbf{u}}|_x\}$) onto $Span\{\mu_{\tilde{\xi}}|_{\varphi_s(x)}\}$ ($Span\{\tilde{\mathbf{u}}|_{\varphi_s(x)}\}$) for all $s \in I$ and all $x \in V$.

Example 2. As an example of the foregoing, take as (V, \mathbf{g}) the four dimensional line element given by

$$ds^2 = (1 + B^2 \rho^2)^2 (dt^2 - d\rho^2 - dz^2) - (1 + B^2 \rho^2)^{-2} \rho^2 d\phi^2, \quad (4.9)$$

where the coordinate ranges are $0 < \rho < \infty$, $-\infty < t, z < \infty$, $0 < \phi < 2\pi$. This is a solution of Einstein-Maxwell equations with a uniform magnetic field $2B$ along the

z -axis known as Melvin solution [42, 11, 51]. Consider the one-parameter group of diffeomorphisms $\varphi_s : \rho \rightarrow \rho + s$ whose generator is the spacelike vector field $\vec{\xi} = \frac{\partial}{\partial \rho}$. The pull-back of the metric tensor under this group turns out to be:

$$\begin{aligned} \varphi_s^* \mathbf{g} &= (1 + B^2(\rho + s)^2)(dt^2 - d\rho^2 - dz^2) - \frac{(\rho + s)^2}{(1 + B^2(\rho + s)^2)^2} d\phi^2 = \\ &= \frac{(1 + B^2(\rho + s)^2)^2}{(1 + B^2\rho^2)^2} ((\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2) - \frac{(\rho + s)^2(1 + B^2\rho^2)^2}{\rho^2(1 + B^2(\rho + s)^2)^2} (\theta^3)^2, \end{aligned}$$

where we have written $\varphi_s^* \mathbf{g}$ in the cobasis of orthonormal 1-forms θ^a (dual to its orthonormal basis of eigenvectors). According to proposition A.2, $\varphi_s^* \mathbf{g} \in \mathcal{DP}_2^+(V)$ if the inequality

$$\frac{(1 + B^2(\rho + s)^2)^2}{\rho + s} \geq \frac{(1 + B^2\rho^2)^2}{\rho}$$

is fulfilled. It is fairly simple to check that this happens if $\rho > 1/\sqrt{3B}$ and $s > 0$ so $\vec{\xi}$ is a CPVF in the region with $\rho > 1/\sqrt{3B}$ whereas $-\vec{\xi}$ is so in $\rho < 1/\sqrt{3B}$. From the formula of $\varphi_s^* \mathbf{g}$ one gets $\text{Span}\{\mu_{\vec{\xi}}\} = \text{Span}\{\partial_t, \partial_\rho, \partial_z\}$.

It is also an easy task to show that $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ adopts the form displayed in theorem 4.1 by just rearranging it a little bit:

$$\begin{aligned} \mathcal{L}_{\vec{\xi}} \mathbf{g} &= 4B^2\rho(1 + B^2\rho^2)(dt^2 - d\rho^2 - dz^2) - \frac{2\rho(1 - B^2\rho^2)}{(1 + B^2\rho^2)^3} d\phi^2 = \alpha \mathbf{g} + \mathbf{U}, \\ \mathbf{U} &= \frac{3B^2\rho^2 - 1}{\rho(1 + B^2\rho^2)} \mathbf{S}, \quad \mathbf{S} = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 + (\theta^3)^2, \quad \alpha = \frac{1}{\rho} \end{aligned}$$

which means that $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ is a linear combination of \mathbf{g} and the Lorentz tensor \mathbf{S} . This is one of the defining features of *nondegenerate bi-conformal vector fields* [22] which, in particular, are the only possible CPVF if $\chi(\mu_{\vec{\xi}})$ is the spacetime dimension minus one (see next proposition).

Proposition 4.2 *Every CPVF with $n - 1$ linearly independent canonical null directions for $n \geq 3$ satisfies*

$$\mathcal{L}_{\vec{\xi}} g_{ab} = \alpha g_{ab} + \beta S_{ab}, \quad \mathcal{L}_{\vec{\xi}} S_{ab} = \alpha S_{ab} + \beta g_{ab},$$

where \mathbf{S} is the Lorentz tensor built up from the set of linearly independent canonical null directions and $\beta > 0$.

Proof : Since the number of linearly independent canonical null directions is greater or equal than 2 we know that $\vec{\xi}$ will satisfy equations (4.1) and (4.2) with $\mu_{\vec{\xi}} = \mu(\mathbf{S})$ so \mathbf{S} is the Lorentz tensor of a simple $n - 1$ form. Therefore the tensor \mathbf{Q} appearing in those equations is a causal tensor with n linearly independent principal directions which means that it is proportional to the metric tensor. Putting $\mathbf{Q} = \gamma \mathbf{g}$ in (4.1) and (4.2) we get:

$$\mathcal{L}_{\vec{\xi}} g_{ab} = \alpha g_{ab} + \beta S_{ab} + \gamma g_{ab}, \quad \mathcal{L}_{\vec{\xi}} S_{ab} = \alpha S_{ab} + \beta g_{ab} + \gamma S_{ab}$$

which is the desired form if we relabel $\alpha \rightarrow \alpha + \gamma$. \square

Nondegenerate bi-conformal vector fields are the main subject of [22] where a thorough study of the properties and applications of these symmetries is performed.

Equation (4.1) can always be written in terms of the Lie derivative of the contravariant metric g^{ab} by using the formula $\mathcal{L}_{\vec{\xi}} g^{ab} = -g^{ap}g^{bq}\mathcal{L}_{\vec{\xi}} g_{pq}$:

$$\mathcal{L}_{\vec{\xi}} g^{ab} = -\alpha g^{ab} - \beta S^{ab} - Q^{ab} \quad (4.10)$$

which is valid regardless of the cases (i)-(iii) of theorem 4.1. Then, formula (4.2) in theorem 4.1 turns out to admit more equivalent forms as we prove next.

Proposition 4.3 *If equation (4.1) of point (i) in theorem 4.1 holds, then equation (4.2) is equivalent to either of the following:*

$$\mathcal{L}_{\vec{\xi}} \vec{\Omega} = \frac{-m}{2}(\alpha + \beta + \lambda)\vec{\Omega}, \quad \mathcal{L}_{\vec{\xi}} \Omega = \frac{m}{2}(\alpha + \beta + \lambda)\Omega, \quad \mathcal{L}_{\vec{\xi}} S^a_b = 0, \quad (4.11)$$

where the m -form Ω gives rise to the Lorentz tensor \mathbf{S} according to the conventions introduced at the end of section 3 (in particular $m = \chi(\mu_{\vec{\xi}}) = \dim(\text{Span}\{\mu(\mathbf{S})\}) > 1$).

Proof : Let us first assume that equations (4.1) and (4.2) are fulfilled. Then equation (4.10) can be used to work out $\mathcal{L}_{\vec{\xi}} S^a_b$ which turns out to be:

$$\mathcal{L}_{\vec{\xi}} S^a_b = \mathcal{L}_{\vec{\xi}} (g^{ap} S_{pb}) = (-\alpha g^{ap} - \beta S^{ap} - Q^{ap}) S_{pb} + g^{ap} (\alpha S_{pb} + \beta g_{pb} + Q_{pc} S^c_b) = 0$$

where the property $Q_{ap} S^p_b = Q_{bp} S^p_a$ has been used. By means of the invariance of S^a_b we can obtain the sought expressions for the Lie derivatives of the normalized forms $\vec{\Omega}$ and Ω as follows: from the algebraic properties of \mathbf{S} any $\vec{k} \in \mu_{\vec{\xi}}$ is a null eigenvector of S^a_b with unit eigenvalue i.e. $S^a_b k^b = k^a$. Taking the Lie derivative of this equation we deduce at once that $S^a_b \mathcal{L}_{\vec{\xi}} k^b = \mathcal{L}_{\vec{\xi}} k^a$ which entails $\mathcal{L}_{\vec{\xi}} \vec{k} \in \text{Span}\{\mu_{\vec{\xi}}\}$. Therefore $\mathcal{L}_{\vec{\xi}} \vec{\Omega} = \mathcal{L}_{\vec{\xi}} (\vec{k}^1 \wedge \dots \wedge \vec{k}^m) = \psi \vec{k}^1 \wedge \dots \wedge \vec{k}^m$ for some proportionality factor to be determined later. Using this, we can work out $\mathcal{L}_{\vec{\xi}} \Omega$ in the usual fashion getting

$$\mathcal{L}_{\vec{\xi}} \Omega = [\psi + m(\alpha + \beta + \lambda)]\Omega, \quad (4.12)$$

where the expression for $\mathcal{L}_{\vec{\xi}} g$ has been used. The function ψ is fixed by Lie derivating the normalization condition $\Omega \cdot \Omega = 2m!(-1)^{m-1}$ yielding $\psi = \frac{-m}{2}(\alpha + \beta + \lambda)$.

Suppose now that either of the equations of (4.11) together with (4.1) hold. Equation (4.2) clearly comes from $\mathcal{L}_{\vec{\xi}} S^a_b = 0$ so it is enough to prove the equivalence of all the equations in (4.11) if (4.1) holds. In the previous paragraph we already proved that $\mathcal{L}_{\vec{\xi}} S^a_b$ implies the formulae involving $\mathcal{L}_{\vec{\xi}} \Omega$ and $\mathcal{L}_{\vec{\xi}} \vec{\Omega}$ and clearly these two equations are equivalent if (4.1) is true. Therefore we only need to show that the first couple of (4.11) implies $\mathcal{L}_{\vec{\xi}} S^a_b = 0$ which again is straightforward using (4.1) (and its consequence (4.10)). \square

Remark. Equations (4.11) are nothing but the infinitesimal formulation of lemma 3.1 and proposition 3.6 and they clearly state the conformal invariance of $\vec{\Omega}$ and Ω as one of the key properties of causal symmetries with a number of linearly independent canonical null directions greater than one. For the case without canonical null directions but *aligned*, so that there is an invariant timelike direction \vec{u} , Eqs.(4.11) still hold with $\Omega = \vec{u}$ and $\mathbf{S} \propto \mathbf{T}\{\vec{u}\}$ the corresponding Lorentz tensor. They are thus equivalent to $\mathcal{L}_{\vec{\xi}} S_{ab} = \alpha S_{ab} + \beta g_{ab} + Q_{ac} S^c_b$ for this Lorentz tensor; see remark (ii) of theorem 4.1.

Theorem 4.1 involves a stability property under repeated action of the differential operator $\mathcal{L}_{\vec{\xi}}$ in a sense which we are going to explain next. The set of principal directions of the tensor \mathbf{U} coincides with the set of canonical null directions of the

submonoid so one might expect that \mathbf{U} satisfied a differential condition similar to that fulfilled by \mathbf{g} . To prove this, we need a preparatory lemma.

Lemma 4.2 *Let $\mathbf{T} \in \mathcal{DP}_2^+(V)$ be a future tensor such that $\mu(\mathbf{T}|_x) \supseteq \mu_{\vec{\xi}}|_x \forall x \in V$ and $\chi(\mu_{\vec{\xi}}) \neq 1$. Then there exists a smooth function ψ such that $\mathcal{L}_{\vec{\xi}}\mathbf{T} - \psi\mathbf{T} \in \mathcal{DP}_2^+(V)$ and $\mu(\mathcal{L}_{\vec{\xi}}\mathbf{T} - \psi\mathbf{T}) \supseteq \mu_{\vec{\xi}}$.*

Proof : The assumption $\mu(\mathbf{T}|_x) \supseteq \mu_{\vec{\xi}}|_x$ implies, according to proposition 3.5, that $\mu(\varphi_s^*\mathbf{T}|_x) = \sigma(\varphi_s^*\mathbf{T}|_x) = \mu_{\vec{\xi}}|_x \forall x \in V, \forall s > 0$. As $\chi(\mu_{\vec{\xi}}) \neq 1$ we can apply lemma A.2 to the future tensors $\varphi_{s_1}^*\mathbf{T}$ and $\varphi_{s_2}^*\mathbf{T}$ and write for every $s_2, s_1 \geq 0$:

$$\varphi_{s_2}^*\mathbf{T} = \alpha_{s_2, s_1} \varphi_{s_1}^*\mathbf{T} + \mathbf{R}_{s_2, s_1},$$

where $\mathbf{R}_{s_2, s_1} \in \mathcal{DP}_2^+(V)$ and α_{s_2, s_1} is a positive function which can always be chosen in such a way that $\alpha_{s_1, s_1} = 1$. This choice implies that $\mathbf{R}_{s_1, s_1} = 0$ so if we let s_1 fixed we can construct the one-parameter family $\mathbf{T}_s \equiv \varphi_s^*\mathbf{T} - \alpha_{s, s_1} \varphi_{s_1}^*\mathbf{T}$ fulfilling the requirements of lemma 4.1 with $s_0 = s_1$ from what we conclude that

$$\left(\frac{d(\varphi_s^*\mathbf{T})}{ds} \Big|_{s=s_1} - \frac{d\alpha_{s, s_1}}{ds} \Big|_{s=s_1} (\varphi_{s_1}^*\mathbf{T}) \right) \in \mathcal{DP}_2^+(V),$$

where $s_1 \in [0, \epsilon)$ (note that this tensor vanishes when acting over any element of $\mu_{\vec{\xi}}$). By putting $s_1 = 0$ in this last equation both results follow. \square

Remark. Note that this lemma holds even if $\mu_{\vec{\xi}} = \emptyset$

Theorem 4.2 *For a nondegenerate CPVF $\vec{\xi}$ we can find smooth functions $\alpha, \alpha_1, \dots, \alpha_r, \dots$ (free functions) such that*

$$\mathbf{Q}_{r+1} \equiv (\mathcal{L}_{\vec{\xi}} - \alpha_r) \cdots (\mathcal{L}_{\vec{\xi}} - \alpha_1) (\mathcal{L}_{\vec{\xi}} - \alpha) \mathbf{g} \in \mathcal{DP}_2^+(V), \forall r \in \mathbb{N}.$$

Furthermore, $\mu(\mathbf{Q}_r) \supseteq \mu_{\vec{\xi}}, \forall r \in \mathbb{N}$.

Proof : The causal tensor \mathbf{U} has $\mu_{\vec{\xi}}$ as its set of principal directions and by assumption $\chi(\mu_{\vec{\xi}}) \neq 1$ so we can apply lemma 4.2 to deduce the existence of a causal tensor \mathbf{Q}_2 and a function α_1 such that $\mathbf{Q}_2 \equiv \mathcal{L}_{\vec{\xi}}\mathbf{U} - \alpha_1\mathbf{U}$. Clearly \mathbf{Q}_2 complies again with lemma 4.2 so we get in the same fashion a causal tensor \mathbf{Q}_3 and a function α_2 such that $\mathbf{Q}_3 = \mathcal{L}_{\vec{\xi}}\mathbf{Q}_2 - \alpha_2\mathbf{Q}_2$ and so on. In this way we obtain a chain of equations

$$\begin{aligned} \mathcal{L}_{\vec{\xi}}\mathbf{g} - \alpha\mathbf{g} &= \mathbf{U} \\ \mathcal{L}_{\vec{\xi}}\mathbf{U} - \alpha_1\mathbf{U} &= \mathbf{Q}_2 \\ \mathcal{L}_{\vec{\xi}}\mathbf{Q}_2 - \alpha_2\mathbf{Q}_2 &= \mathbf{Q}_3 \\ &\dots\dots\dots \\ \mathcal{L}_{\vec{\xi}}\mathbf{Q}_r - \alpha_r\mathbf{Q}_r &= \mathbf{Q}_{r+1} \\ &\dots\dots\dots, \end{aligned} \tag{4.13}$$

where the tensors on the right hand side of each equation belong to $\mathcal{DP}_2^+(V)$ and contain $\mu_{\vec{\xi}}$ as a subset of its set of principal directions due to lemma 4.2. The theorem follows because $\mathbf{Q}_r = (\mathcal{L}_{\vec{\xi}} - \alpha_{r-1})(\mathcal{L}_{\vec{\xi}} - \alpha_{r-2}) \cdots (\mathcal{L}_{\vec{\xi}} - \alpha_1)(\mathcal{L}_{\vec{\xi}} - \alpha)\mathbf{g}$. \square

The chain of equations (4.13) used in the proof of this theorem can be reduced to a finite number by noticing that we must reach an index j for which the causal tensor

\mathbf{Q}_{j+1} can be written in terms of the previous causal tensors as $\mathbf{Q}_{j+1}|_x = \phi_0(x)\mathbf{g}|_x + \phi_1(x)\mathbf{U}|_x + \sum_{k=2}^j \phi_k(x)\mathbf{Q}_k|_x$ for some smooth functions $\phi_0(x), \phi_1(x), \phi_2(x), \dots, \phi_j(x)$. Thus the last equation of the chain can be taken to be, by redefining α_j if necessary

$$\mathcal{L}_{\tilde{\xi}} \mathbf{Q}_j - \alpha_j \mathbf{Q}_j = \phi_0 \mathbf{g} + \phi_1 \mathbf{U} + \sum_{k=2}^{j-1} \phi_k \mathbf{Q}_k. \quad (4.14)$$

Up to now we have provided a thorough study of the properties of CPVFs and we have found the necessary conditions required for a vector field to be causal preserving. However, we still have no means to decide whether a given vector field is causal preserving or not. Next, we solve this problem and show that the necessary conditions of theorem 4.1 are in fact sufficient by proving its converse.

Theorem 4.3 *Let $\tilde{\xi}$ be a smooth complete vector field and assume that there exists a function α such that $\mathbf{R} \equiv \mathcal{L}_{\tilde{\xi}} \mathbf{g} - \alpha \mathbf{g} \in \mathcal{DP}_2^+(V)$ with \mathbf{R} keeping the same number of principal null directions on V .*

- If $\mu(\mathbf{R}) = \emptyset$ then $\tilde{\xi}$ is a CPVF with $\mu_{\tilde{\xi}} = \emptyset$.
- If $\mu(\mathbf{R}) \neq \emptyset$ and $\mathcal{L}_{\tilde{\xi}} \Omega \propto \Omega$ where Ω is a non-zero m -form of maximum degree on $\text{Span}\{\mu(\mathbf{R})\}$, then $\tilde{\xi}$ is a CPVF with $\mu_{\tilde{\xi}} = \mu(\mathbf{R})$.

Proof : We will perform the proof separating out the cases with $\mu(\mathbf{R}) \neq \emptyset$ and $\mu(\mathbf{R}) = \emptyset$. Assume first that the causal tensor \mathbf{R} has no principal directions. This means that, for every null vector $\vec{\mathbf{k}}$, $\mathcal{L}_{\tilde{\xi}} \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) > 0 \ \forall x \in V$ so that the function $f_{\vec{\mathbf{k}}}(s) \equiv \varphi_s^* \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}})$ is positive for all $s \in (0, \epsilon)$ and negative in $(-\epsilon', 0)$ for some small $\epsilon, \epsilon' > 0$. In fact, the result holds for all positive values of ϵ', ϵ because, if it did not, there would be a first value $s_0 > 0$ (the proof for negative values of s is analogous) such that $0 = f_{\vec{\mathbf{k}}}(s_0) = \mathbf{g}|_{\varphi_{s_0}(x)}(\varphi'_{s_0} \vec{\mathbf{k}}, \varphi'_{s_0} \vec{\mathbf{k}})$ implying that $\varphi'_{s_0} \vec{\mathbf{k}}$ is a null vector so that, using our assumption again, $\mathcal{L}_{\tilde{\xi}} \mathbf{g}|_{\varphi_{s_0}(x)}(\varphi'_{s_0} \vec{\mathbf{k}}, \varphi'_{s_0} \vec{\mathbf{k}})$ should also be strictly positive. Thus we could write $0 < \mathcal{L}_{\tilde{\xi}} \mathbf{g}|_{\varphi_{s_0}(x)}(\varphi'_{s_0} \vec{\mathbf{k}}, \varphi'_{s_0} \vec{\mathbf{k}}) = (\varphi_{s_0}^* \mathcal{L}_{\tilde{\xi}} \mathbf{g})|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = d\varphi_s^* \mathbf{g}|_x/ds|_{s=s_0}(\vec{\mathbf{k}}, \vec{\mathbf{k}})$ where in the last step the standard formula

$$\frac{d\varphi_s^* \mathbf{T}}{ds} = \varphi_s^* \mathcal{L}_{\tilde{\xi}} \mathbf{T}, \quad (4.15)$$

must be used. Hence, $f_{\vec{\mathbf{k}}}(s)$ would be strictly increasing at $s = s_0$ too, which is obviously a contradiction.

If $\mu(\mathbf{R}) \neq \emptyset$ then this set consists of the future-pointing null vector fields $\vec{\mathbf{k}} \in T(V)$ such that $\mathcal{L}_{\tilde{\xi}} \mathbf{g}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$. Let us show that these null vectors remain in $\mu(\mathbf{R})$ under the push-forward of φ_s , $\forall s \in \mathbb{R}$ (and hence the equality $\mathcal{L}_{\tilde{\xi}} \mathbf{g}(\varphi'_s \vec{\mathbf{k}}, \varphi'_s \vec{\mathbf{k}}) = 0$ holds $\forall s \in \mathbb{R}$ on the whole V). Recall that we assume that the number $m = \text{Span}\{\mu(\mathbf{R})\}$ does not depend on the point of V . The result is evident if $m = 1$, given that then $\Omega \propto \mathbf{k}$ with $\vec{\mathbf{k}}$ the unique direction in $\mu(\mathbf{R})$ so that the hypothesis is $\mathcal{L}_{\tilde{\xi}} \mathbf{k} \propto \mathbf{k}$ which easily leads to $\varphi'_s \vec{\mathbf{k}} \propto \vec{\mathbf{k}}$ all over V . Thus, we will concentrate on the case with $m > 1$. Define the m -form $\Omega = \mathbf{k}_1 \wedge \dots \wedge \mathbf{k}_m$ built up from m linearly independent elements of $\mu(\mathbf{R})$ and construct its Lorentz tensor \mathbf{S} . The canonical decomposition of \mathbf{R} (see theorem A.1) reads $\mathbf{R} = \beta \mathbf{S} + \mathbf{Q}$ with $\beta > 0$, $\mathbf{Q} \in \mathcal{DP}_2^+(V)$, $\mu(\mathbf{Q}) \supset \mu_{\tilde{\xi}}$ and $\mu(\mathbf{S}) = \mu(\mathbf{R})$, so that (4.1) holds. Then, we know from proposition 4.3 that the property $\mathcal{L}_{\tilde{\xi}} \Omega \propto \Omega$ is equivalent to equation (4.2). Since equations (4.1) and (4.2) lead to (4.8), we obtain

via integration the following formula for the projector $\mathbf{P} = (\mathbf{g} + \mathbf{S})/2$ over the subspace $\text{Span}\{\mu(\mathbf{S})\} = \text{Span}\{\mu(\mathbf{R})\}$

$$\mathcal{L}_{\vec{\xi}} \mathbf{P} = (\alpha + \beta + \lambda) \mathbf{P} \implies (\varphi_s^* \mathbf{P}) \propto \mathbf{P}, \quad (4.16)$$

where again equation (4.15) has been used. Obviously, $\mathbf{S}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0 \iff \mathbf{P}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$, whence $\forall \vec{\mathbf{k}} \in \mu(\mathbf{R}|_x)$, $\mathbf{P}|_{\varphi_s(x)}(\varphi'_s \vec{\mathbf{k}}, \varphi'_s \vec{\mathbf{k}}) = \varphi_s^* \mathbf{P}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \propto \mathbf{P}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$, which is only possible if either $\varphi'_s \vec{\mathbf{k}}$ belongs to $\perp \text{Span}\{\mu(\mathbf{S}|_{\varphi_s(x)})\}$ or $\varphi'_s \vec{\mathbf{k}} \in \text{Span}\{\mu(\mathbf{S}|_{\varphi_s(x)})\}$ i.e. $\mathcal{L}_{\vec{\xi}} \mathbf{g}|_{\varphi_s(x)}(\varphi'_s \vec{\mathbf{k}}, \varphi'_s \vec{\mathbf{k}}) = 0$ for all $s \in \mathbb{R}$. The former possibility is forbidden because the conformal invariance $\mathcal{L}_{\vec{\xi}} \Omega \propto \Omega$ implies, via integration by means of (4.15), that $\varphi'_s \vec{\mathbf{k}}|_{\varphi_s(x)} \in \text{Span}\{\vec{\mathbf{k}}_1|_{\varphi_s(x)}, \dots, \vec{\mathbf{k}}_m|_{\varphi_s(x)}\} = \text{Span}\{\mu(\mathbf{S}|_{\varphi_s(x)})\}$. The conclusion of all this is that $0 = \mathcal{L}_{\vec{\xi}} \mathbf{g}|_{\varphi_s(x)}(\varphi'_s \vec{\mathbf{k}}, \varphi'_s \vec{\mathbf{k}}) = d\varphi_s^* \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}})/ds$, $\forall \vec{\mathbf{k}} \in \mu(\mathbf{R}|_x)$ and every $s \in \mathbb{R}$ which means that, all over V , $\varphi_s^* \mathbf{g}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0 \forall s \in \mathbb{R}$ and $\forall \vec{\mathbf{k}} \in \mu(\mathbf{R})$. If, on the other hand, $\vec{\mathbf{k}}$ is a null vector such that $\mathcal{L}_{\vec{\xi}} \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) > 0$, a reasoning similar to that of the case with $\mu(\mathbf{R}) = \emptyset$ leads us to $\varphi_s^* \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) > 0$ if $\vec{\mathbf{k}} \notin \mu(\mathbf{R}|_x)$ and $s > 0$, for all $x \in V$ (note that we must use the just proven property of the principal directions in order to be able to show the inequality $\mathcal{L}_{\vec{\xi}} \mathbf{g}|_{\varphi_s(x)}(\varphi'_s \vec{\mathbf{k}}, \varphi'_s \vec{\mathbf{k}}) > 0 \forall s > 0$ if $\vec{\mathbf{k}} \notin \mu(\mathbf{R}|_x)$).

In any case, we end up proving that, for every $x \in V$, $\varphi_s^* \mathbf{g}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \geq 0$ for all $s > 0$ and $\forall \vec{\mathbf{k}} \in \partial\Theta_x^+$, which according to theorem A.3 is equivalent to $\varphi_s \in \mathcal{C}(V, \mathbf{g}) \forall s > 0$. Furthermore, we get $\mu(\mathbf{R}) = \mu_{\vec{\xi}}$, as desired. \square

Remark. We must point out that the requirement $\mathcal{L}_{\vec{\xi}} \mathbf{g} - \alpha \mathbf{g} \in \mathcal{DP}_2^+(V)$ can be replaced by the equivalent condition $\mathcal{L}_{\vec{\xi}} \mathbf{g}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) > 0$ for every null vector field if $\mu(\mathbf{R}) = \emptyset$, that is to say, the tensor $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ fulfills the strict null convergence condition (see lemma A.3 for a proof of this). This last condition is more natural as it only involves the tensor $\mathcal{L}_{\vec{\xi}} \mathbf{g}$. For the case $\mu(\mathbf{R}) \neq \emptyset$ we have preferred to formulate theorem 4.3 in terms of the dominant property of the tensor $\mathcal{L}_{\vec{\xi}} \mathbf{g} - \alpha \mathbf{g}$ for a certain function α , but equivalent statements using the null convergence condition of $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ can also be given, see the remark after lemma A.3.

Theorem 4.3 covers all possibilities for a vector field $\vec{\xi}$ liable to generate a one-parameter submonoid of causal symmetries according to the number of canonical null directions. This theorem together with theorem 4.1 comprise the necessary and sufficient conditions which a vector field must meet in order to be causal preserving. Notice that we have made no hypotheses over the differentiability properties of the metric tensor in theorem 4.3 whereas the analyticity of the metric was essential to state theorem 4.1.

A proof of theorem 4.3 for the particular case in which $\vec{\xi}$ is timelike is available in [31, 30] where some additional global properties of such vector fields are discussed. These authors require the null convergence condition for the tensor $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ in order to prove that a timelike $\vec{\xi}$ is a CPVF which, as we have already pointed out, is equivalent to the condition $\mathcal{L}_{\vec{\xi}} \mathbf{g} - \alpha \mathbf{g} \in \mathcal{DP}_2^+(V)$ demanded in theorem 4.3 if $\mathcal{L}_{\vec{\xi}} \mathbf{g}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) > 0$ for every null vector $\vec{\mathbf{k}}$. As a matter of fact, the proof in [31, 30] does not depend on the timelike character of $\vec{\xi}$ in this case. However, for the general case with canonical null directions, i.e. such that the null convergence condition for $\mathcal{L}_{\vec{\xi}} \mathbf{g}$ is not strict, their proof can only be applied to timelike CPVFs, as they clearly point out, whereas the result stated in theorem 4.3 is thoroughly general and covers all possibilities.

Theorem 4.3 states in particular that the canonical null directions of the one-parameter submonoid generated by $\vec{\xi}$ are just $\mu_{\vec{\xi}} = \{\vec{k} \in \partial\Theta^+ : \mathcal{L}_{\vec{\xi}}\mathbf{g}(\vec{k}, \vec{k}) = 0\}$. In addition, we can get the following interesting corollary.

Corollary 4.1 *If the vector fields $\vec{\xi}_1$ and $\vec{\xi}_2$ are causal preserving and $\mu_{\vec{\xi}_1} = \mu_{\vec{\xi}_2}$, then every linear combination of them with positive constant coefficients is causal preserving too with the same canonical null directions.*

Proof : This is a consequence of the fact that every linear combination $\vec{\xi} = c_1\vec{\xi}_1 + c_2\vec{\xi}_2$ with $c_1, c_2 > 0$ also satisfies the differential conditions of theorem 4.1 whenever $\mu_{\vec{\xi}_1} = \mu_{\vec{\xi}_2}$, and the fact that theorem 4.3 proves the sufficiency of these equations for $\vec{\xi}$. \square

5. Conserved quantities and constants of motion

We explore next the possibility of defining conserved quantities from CPVFs in much the same way as it has been done with other known symmetries such as isometries or conformal motions. For a general CPVF $\vec{\xi}$ we already know (see (4.5)) that $\mathcal{L}_{\vec{\xi}}\mathbf{g} = \alpha\mathbf{g} + \mathbf{U}$ for some $\mathbf{U} \in \mathcal{DP}_2^+$. Then, if \vec{v} is the vector tangent to a given curve, we get along the curve

$$\begin{aligned} \nabla_{\vec{v}}(\vec{\xi} \cdot \vec{v}) &= (\nabla_{\vec{v}}\vec{\xi}) \cdot \vec{v} + \vec{\xi} \cdot (\nabla_{\vec{v}}\vec{v}) = \frac{1}{2}(\mathcal{L}_{\vec{\xi}}\mathbf{g})(\vec{v}, \vec{v}) + \vec{\xi} \cdot (\nabla_{\vec{v}}\vec{v}) = \\ &= \frac{1}{2}\alpha\mathbf{g}(\vec{v}, \vec{v}) + \frac{1}{2}\mathbf{U}(\vec{v}, \vec{v}) + \vec{\xi} \cdot \nabla_{\vec{v}}\vec{v}. \end{aligned}$$

From this relation we deduce that if \vec{v} is tangent to an affinely parametrized geodesic such that $\vec{v}|_x \in \mu(\mathbf{U}|_x) = \mu_{\vec{\xi}}|_x$ for all x on the curve, then $\vec{\xi} \cdot \vec{v}$ is constant along this null geodesic. Hence, the null geodesics along the canonical null directions of a CPVF $\vec{\xi}$ have a *constant* component along $\vec{\xi}$. Furthermore, for general affinely parametrized null geodesics it follows that $\vec{\xi} \cdot \vec{v}$ is monotonically non-decreasing to the future along the curve. However, we would like to consider a class of curves more general than just null geodesics. The next definition taken from [48] will help us in this task.

Definition 5.1 *A smooth curve $\gamma \subset V$ is called a subgeodesic with respect to the vector field \vec{p} if its tangent vector satisfies the equation $\nabla_{\vec{v}}\vec{v} = \vec{p}\lambda + a\vec{v}$ where λ is a smooth function.*

We can always choose an *affine parametrization* for every subgeodesic so that the term $a\vec{v}$ is removed from previous equation: $\nabla_{\vec{v}}\vec{v} = \vec{p}\lambda$. We will assume in the sequel that we are working with affinely parametrized subgeodesics.

Next we show how to define constants along certain of these curves.

Proposition 5.1 *Let \vec{v} be the tangent vector of an affinely parametrized subgeodesic with respect to \vec{p} . Then for a CPVF $\vec{\xi}$, the quantity $\vec{\xi} \cdot \vec{v}$ is constant along the subgeodesic if either:*

- (i) $\vec{\xi}$ and \vec{p} are orthogonal at all points of the curve and $\mathbf{U}(\vec{v}, \vec{v}) = -\alpha\vec{v} \cdot \vec{v}$.
- (ii) $\lambda = -\frac{1}{2}(\vec{\xi} \cdot \vec{p})^{-1}(\alpha\mathbf{g} + \mathbf{U})(\vec{v}, \vec{v})$ if $\vec{\xi}$ and \vec{p} are not orthogonal at some point of the subgeodesic.

Proof : We just replace $\nabla_{\vec{v}}\vec{v}$ in the above calculation of $\nabla_{\vec{v}}(\vec{\xi} \cdot \vec{v})$ by its expression for an affinely parametrized subgeodesic, getting:

$$\nabla_{\vec{v}}(\vec{\xi} \cdot \vec{v}) = \frac{1}{2}\alpha\mathbf{g}(\vec{v}, \vec{v}) + \frac{1}{2}\mathbf{U}(\vec{v}, \vec{v}) + \lambda\vec{\xi} \cdot \vec{p},$$

from what the result for each case (i) and (ii) follows. \square

Remarks.

- (i) Conditions (i) and (ii) are just necessary conditions imposed upon the CPVF $\vec{\xi}$ and the vector \vec{p} for $\vec{\xi} \cdot \vec{v}$ to be a constant.
- (ii) Case (i) of the previous proposition includes the case $\vec{v} \in \mu(\mathbf{U}) = \mu_{\vec{\xi}}$. Observe that λ is arbitrary in this case, so that for $\lambda = 0$ we can have the null geodesics mentioned before.
- (iii) The condition in (ii) is very weak because *every* vector field \vec{p} non-orthogonal to $\vec{\xi}$ gives rise to a constant of motion along the affine subgeodesics with respect to \vec{p} by simply choosing an adequate λ proportional to $(\alpha\mathbf{g} + \mathbf{U})(\vec{v}, \vec{v})$ (this is similar to the case studied in [16]).

The construction of conserved currents is also possible for CPVFs as we prove in the next proposition.

Proposition 5.2 *Let $T_{a_1 \dots a_p}$ be a totally symmetric and traceless rank- p tensor and assume further that $\nabla_b T_{a_2 \dots a_p}^b = 0$. Then for a CPVF $\vec{\xi}$ the current $j^b = T_{a_2 \dots a_p}^b \xi^{a_2} \dots \xi^{a_p}$ is divergence-free if $T_{bca_3 \dots a_p} U^{bc} = 0$.*

Proof : A direct calculation using the assumptions on $T_{a_1 \dots a_p}$ gives

$$\nabla_b j^b = (p-1)T^{bc}{}_{a_3 \dots a_p} \nabla_{(b} \xi_{c)} \xi^{a_3} \dots \xi^{a_p} = \frac{p-1}{2} T^{bc}{}_{a_3 \dots a_p} U_{bc} \xi^{a_3} \dots \xi^{a_p}$$

where (4.5) has been used for $2\nabla_{(b} \xi_{c)} = \mathcal{L}_{\vec{\xi}} \mathbf{g}_{bc}$. The result is now trivial. \square

Observe that the condition of the proposition is sufficient, but it will be enough with just $T^{bc}{}_{a_3 \dots a_p} U_{bc} \xi^{a_3} \dots \xi^{a_p} = 0$.

Relevant examples of tensors with the properties stated in proposition 5.2 spring to mind at once as for instance every traceless Einstein tensor, the energy-momentum tensor of a source-free electromagnetic field in four dimensions, perfect fluids whose equation of state is $p = \rho/(n-1)$, and many superenergy tensors [49] like the Bel-Robinson tensor in Einstein spaces [3] or the trace of the Chevreton tensor for electromagnetic fields [8]. In any case one must check the condition $T_{ab \dots a_p} U^{ab} = 0$ in order to actually obtain a conserved current which in general is not a trivial matter and requires a case-by-case discussion. We present next some examples of physical relevance.

The electromagnetic field

The simplest tensor one can think of complying with the above conditions is the energy-momentum tensor of any source-free electromagnetic field in four dimensions. This is a traceless rank-two tensor T_{ab} which is divergence-free if Maxwell equations are satisfied and hence $j^a = T^a_b \xi^b$ will be conserved for a CPVF $\vec{\xi}$ if $T_{ab} U^{ab} = 0$.

When the electromagnetic field is null then $T_{ab} = k_a k_b$ where \mathbf{k} is the single principal null direction of the Maxwell field. Therefore, $T_{ab} U^{ab}$ will vanish if $\vec{k} \in \mu_{\vec{\xi}}$ or,

in other words, if $\vec{\mathbf{k}}$ is a canonical null direction of the submonoid of causal symmetries generated by $\vec{\xi}$ (note that in this case T_{ab} is traceless for any dimension n).

Another possibility arises when U_{ab} is proportional to a Lorentz tensor S_{ab} commuting with T_{ab} , and the electromagnetic field is non-null. The square of the (symmetric) tensor $T_{ap}U_b^p$, in the sense of proposition A.4, is proportional to the metric tensor —due to the Rainich conditions for T_{ab} in four dimensions, see e.g. PI—, whence theorem 6.1 of PI implies that $T_{ap}U_b^p$ must be proportional to the superenergy tensor of a simple form. As its trace vanishes owing to the condition $T_{ab}U^{ab} = 0$, the generalized Rainich conditions of PI tell us that the simple form must be a 2-form from what we deduce, working in the common orthonormal basis of both T_{ab} and U_{ab} , that U_{ab} is also the superenergy tensor of a simple 2-form, different from the ones generating T_{ab} and $T_{ap}U_b^p$. Writing $U_{ab} = \beta S_{ab}$ it is easy to prove by Lie derivating the relation $g_{ab} = S_{ap}S_b^p$ that the vector $\vec{\xi}$ is a bi-conformal vector field (see just before proposition 4.2). Summarizing, the current $j^b = T^b_c \xi^c$ will be conserved if $\vec{\xi}$ is a bi-conformal vector field with $\mu_{\vec{\xi}} \neq \mu(\mathbf{T})$ but such that $\text{Span}\{\mu_{\vec{\xi}}\} \cap \text{Span}\{\mu(\mathbf{T})\}$ defines a common timelike eigendirection for \mathbf{T} and \mathbf{U} .

The Bel-Robinson tensor

Another example of tensor satisfying the main hypotheses of proposition 5.2 is the Bel-Robinson tensor in an Einstein space ($n = 4$). This tensor was introduced independently by Bel [3] and Robinson [47] in General Relativity as a gravitational analog of the energy-momentum tensor of the electromagnetic field and its expression reads

$$\mathcal{T}_{abcd} = C_{apbq}C_c{}^p{}_d{}^q + (*C)_{apbq}(*C)_c{}^p{}_d{}^q,$$

where $*C$ stands for the (unique) dual of the Weyl tensor C_{abcd} . The Bel-Robinson tensor is a completely symmetric, traceless and divergence-free *future* tensor [45, 49]. Generalizations of this construction to the so-called superenergy tensors are presented in [49, 46] (actually one can build from these objects more tensors satisfying the requirements of proposition 5.2). The remaining condition $\mathcal{T}_{abcd}U^{ab} = 0$ of proposition 5.2 can be achieved depending on how the future tensor \mathbf{U} , or equivalently $\mu_{\vec{\xi}}$, is related to the set $\mu(\mathcal{T})$ which, as is well known, consists of the principal null vectors of the Weyl tensor (see e.g. [45, 7]). Several possibilities leading to a conservation law are:

- If $U_{ab} = k_{(a}u_{b)}$ with k_a null and u_a timelike, so that $\mu(\mathbf{U}) = \mu_{\vec{\xi}} = \{\lambda\vec{\mathbf{k}}\}$, $\lambda > 0$ if $\vec{\mathbf{k}}$ is future pointing and $\vec{\xi}$ is a degenerate CPVF. In this case the condition becomes

$$\mathcal{T}_{abcd}U^{ab} = 0 \Rightarrow \mathcal{T}_{abcd}k^a u^b = 0,$$

which entails both $0 = \mathcal{T}_{abcd}k^a$ and the spacetime being of Petrov type N (theorem 2 of [7]) with $\vec{\mathbf{k}}$ as the unique repeated principal null direction of the Weyl tensor. This holds true, in fact, for all causal CPVFs if the set of its canonical null directions includes the principal null direction of the Petrov type N Weyl tensor, because then $\mathcal{T}_{abcd} \propto k_a k_b k_c k_d$ and obviously $\mathcal{T}_{abcd}U^{ab} = 0$. Thus, in any Petrov type N spacetime with $\vec{\mathbf{k}}$ as the repeated principal null direction, all causal CPVFs $\vec{\xi}$ such that $\mu_{\vec{\xi}} \ni \vec{\mathbf{k}}$ provide conserved currents $\mathcal{T}^a{}_{bcd}\xi^b\xi^c\xi^d$.

- Another possibility is that $U_{ab} = k_a k_b$ with $\vec{\mathbf{k}}$ null, so that $\vec{\xi}$ is a degenerate CPVF. In this case the condition becomes $\mathcal{T}_{abcd}k^a k^b = 0$, which again implies

that $\vec{\mathbf{k}}$ is a repeated null direction of the Weyl tensor but now the spacetime can also be of Petrov type III (apart from type N), see [7].

Example 3. In this example we present a spacetime (V, \mathbf{g}) where a nontrivial conserved current can be explicitly constructed in the fashion described above. The line element is:

$$ds^2 = \frac{2}{x^2} \left[(az^2 + b)dt^2 - \frac{dz^2}{az^2 + b} - \frac{dx^2}{x^2(-a + lx - 2e^2x^2)} - x^2(-a + lx - 2e^2x^2)dy^2 \right]$$

where a, b, e, l are constants and $a > 0$. This is a type-D Einstein-Maxwell solution [39, 51] whose energy-momentum tensor takes the form ($8\pi G = c = 1$)

$$T^a_b = e^2 x^4 \text{diag}(1, 1, -1, -1)$$

so that e can be interpreted as the charge creating the electromagnetic field.

The one-parameter group of diffeomorphisms generated by $\xi = \partial/\partial z$ contains a submonoid of proper causal symmetries in the half space $z \geq 0$ because, for $s > 0$:

$$\mathcal{DP}_2^+(V) \ni \varphi_s^* \mathbf{g} = \left[\frac{a(z+s)^2 + b}{az^2 + b} - 1 \right] (\theta^0)^2 + \left[1 - \frac{az^2 + b}{a(z+s)^2 + b} \right] (\theta^1)^2 + \mathbf{g},$$

where we have used the orthonormal cobasis dual to the set of eigenvectors of T^a_b . Therefore we have for the Lie derivative of \mathbf{g}

$$\mathcal{L}_\xi \mathbf{g} = \mathbf{U} = \left(\frac{2az}{az^2 + b} \right) [(\theta^0)^2 + (\theta^1)^2]$$

so that this is an *aligned* CPVF with $\mu_\xi = \emptyset$. We can check that $U^{ab}T_{ab} = 0$ so that the vector field $j^a = T^a_b \xi^b$ is divergence-free. The explicit computation of such current gives $\vec{\mathbf{j}} = e^2 x^4 \partial/\partial z$.

6. Conclusions

We have found the necessary and sufficient conditions for a vector field to be causal preserving, that is to say, to locally generate a one-parameter submonoid—it cannot be a group unless in the known case of conformal Killing vectors—of causal transformations. This implies that the causal future-directed vector fields are preserved along the flow of the causal preserving vector field. These transformations and its infinitesimal generators generalize straightforwardly the corresponding conformal counterparts. The generalization arises because the new vector fields preserve the whole solid Lorentz cone but *only part* of the null cone remains on the null cone. This part is the “conformal” part of the transformation, so to speak. Thus, a classification of causal preserving vector fields can be immediately performed, going from the conformal Killing vectors (the full null cone is preserved), passing through the case where only one null direction is not preserved—these are a particular case of bi-conformal vector fields, see [22]—, and so on until one reaches the so-called degenerate case for which only one null direction is preserved—including the previously studied Kerr-Schild symmetries [16]— and ending with the case where no null directions are kept; in this last case, there may still be one timelike direction which is maintained invariant—the aligned case.

Causal preserving vector fields seem to have relevant applications. In the paper we have presented a few. To start with, they provide constants of motion along preferred null geodesics, or more generally, along subgeodesics. And if not, they give monotonically increasing quantities along the mentioned curves. The importance of subgeodesics in physical applications may arise if a canonical, preferred, or otherwise distinguished, global vector field is defined. For instance, one can choose a particular observer defined by a timelike vector field \vec{u} . Then, obviously, subgeodesics relative to \vec{u} are physically meaningful as they can be seen as autoparallel curves of a (non-metric) connection associated to this observer. As a matter of fact, the so-called chorodesics, see e.g. [5, 6] and references therein, are just a particular case of subgeodesics. For other applications of subgeodesics the reader is referred to [48]. The physical or geometric interpretation of these constants of motion is similar to that arising for Killing vectors: the component along the CPVF of the vector tangent to the (sub)geodesic is constant along the curve. More generally, if these components are not constant, in many cases one can derive monotonicity properties for them, a result which may be relevant in applications of causality theory or global geometry.

Furthermore, the CPVF can be used to construct divergence-free vector fields, currents in short, in a manner analogous to that of conformal Killing or Killing vectors. The importance of these currents, of course, is that they provide conserved quantities for the field involved which are in principle independent of those previously known. We have presented several particular cases for the electromagnetic field or the Bel-Robinson gravitational “superenergy”. The true relevance of these quantities is still to be investigated.

Finally, from a pure geometrical viewpoint, the causal preserving vector fields can be used to characterize splitting of the Lorentzian manifolds—such as in the case of warped product spacetimes and the like— or other generalizations of conformal relations [21, 30, 31, 22], as well as to analyze further the causal structure and isocausality of Lorentzian manifolds in the sense of [20] by using infinitesimal techniques. This is under current study.

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Appendix A: Essentials of causal relationship and causal tensors

We review next the basic tools presented in PII and PI (refs.[20, 10]) which are needed in this paper. A vector \vec{v} in the tangent space of each point of a Lorentzian manifold is classified as spacelike or causal according to whether the sign of the scalar product $\mathbf{g}(\vec{v}, \vec{v})$ is negative or not and causal vectors are further divided into null ($\mathbf{g}(\vec{v}, \vec{v}) = 0$) and timelike ($\mathbf{g}(\vec{v}, \vec{v}) > 0$) ones. If we choose a causal vector $\vec{v}_1 \in T_x(V)$ to be future pointing, the remaining causal vectors are termed as future directed with respect to \vec{v}_1 if $\mathbf{g}(\vec{v}, \vec{v}_1) > 0$ and past directed otherwise. Of course this set remains the same if we replace \vec{v}_1 by another future directed vector \vec{v}_2 with respect to \vec{v}_1 making thus possible to define the set of causal future directed vectors which, following PII, will be denoted by Θ_x^+ . This set has also a past counterpart Θ_x^- (we will omit the past

duals, denoted in general with a “-”, when we define and use other causal objects in this paper). The boundary $\partial\Theta_x^+$ of Θ_x^+ is the set of null future directed vectors.

These definitions extend straightforwardly to the bundle $T(V)$ and so we can speak of spacelike and future-directed causal vectors over V with the notation $\Theta^+(V)$ for the latter set. The set $\Theta(V)$ is the union of $\Theta^+(V)$ and $\Theta^-(V)$. A Lorentzian manifold is causally orientable if it admits an everywhere continuous future-directed vector field (i.e. a section of $\Theta^+(V)$). We will assume that all our Lorentzian manifolds are causally orientable. We will implicitly assume that the causal vectors are future-directed unless otherwise stated.

The previous construction can be repeated for 1-forms if we work with the contravariant form g^{ab} of the metric tensor. Furthermore we can even extend it to higher rank tensors by means of the following definition [49, 10]

Definition A.1 A tensor $\mathbf{T} \in T_r^0(x)$ is said to be future (respectively past) if $\mathbf{T}(\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r) \geq 0$ (resp. ≤ 0) $\forall \vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r \in \Theta_x^+$. Future and past tensors are called causal tensors.

The set of future tensors at x will be denoted by $\mathcal{DP}_r^+|_x$ with the obvious notation for causal tensors over the whole manifold. We will sometimes omit the set over which we are considering causal tensors in order to alleviate the notation. The symbol $(\mathcal{DP}_r^+)_r^s$ will be used for future tensors defined on $T_r^s(V)$ and \mathcal{DP}^+ stands for the set of all future tensors. As was proved in PI, definition A.1 can be equivalently stated by (i) just demanding that $\mathbf{T}(\vec{\mathbf{k}}_1, \dots, \vec{\mathbf{k}}_r) \geq 0$ for all future-directed *null* vectors $\vec{\mathbf{k}}_1, \dots, \vec{\mathbf{k}}_r$ or (ii) requiring that $\mathbf{T}(\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r) > 0$ for all future-directed *timelike* vectors $\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r$. Other interesting criteria to ascertain when a given tensor is in \mathcal{DP}_r^+ are given in [49, 10, 20], as well as some interesting properties of these tensors. We can think of a causal tensor as a higher rank generalization of the tensors satisfying the well known dominant energy condition used in General Relativity (future tensors are also called *dominant tensors*).

We present next a definition (PII) which will play a very important role along this work.

Definition A.2 The principal directions of $\mathbf{T} \in \mathcal{DP}_r^+|_x$ are the vectors $\vec{\mathbf{u}} \in \Theta_x^+$ such that $\mathbf{T}(\vec{\mathbf{u}}, \dots, \vec{\mathbf{u}}) = 0$. The set of principal directions of \mathbf{T} is denoted by $\mu(\mathbf{T})$.

Of course we can define the principal directions for arbitrary elements of $\mathcal{DP}_r^+(V)$ (or its sections) which will consist of elements of $\Theta^+(V)$. Clearly, for an element $\mathbf{T} \in \mathcal{DP}_r^+(V)$, the set of its principal directions may change from point to point and if \mathbf{T} is a continuous (differentiable) section of $\mathcal{DP}_r^+(V)$ we will be able to construct continuous (differentiable) vector fields $\vec{\mathbf{v}}$ such that $\vec{\mathbf{v}}|_x \in \mu(\mathbf{T}|_x)$, $\forall x \in V$. As we are going to show, these principal directions have a great deal to do with the algebraic classification of the tensor under study.

To start with, the principal directions are necessarily null ($i_{\vec{\mathbf{k}}}$ stands for the usual inner contraction with $\vec{\mathbf{k}}$).

Proposition A.1 A vector $\vec{\mathbf{k}} \in \Theta_x^+$ is in $\mu(\mathbf{T})$ for some $\mathbf{T} \in \mathcal{DP}_r^+|_x$, ($r > 1$) iff it is null and belongs to $\mu(i_{\vec{\mathbf{k}}}\mathbf{T})$. In the case of $r = 1$ the condition is $\vec{\mathbf{k}} \propto \mathbf{T}$.

Proof : Clearly $\vec{\mathbf{k}} \in \mu(\mathbf{T}) \Leftrightarrow \vec{\mathbf{k}} \in \mu(i_{\vec{\mathbf{k}}}\mathbf{T})$. On the other hand if $\vec{\mathbf{k}}$ is in $\mu(\mathbf{T})$ then the future-directed 1-form $v_a \equiv T_{ab_2 \dots b_r} k^{b_2} \dots k^{b_r}$ is orthogonal to k^a which is only possible if k^a and v_a are null and proportional (see property 2.2 of PI for an explanation

about the causal character of v_a). When $r = 1$ then $\vec{\mathbf{k}} \in \mu(\mathbf{T}) \Leftrightarrow k^a T_a = 0$ which again only happens if both \mathbf{T} and \mathbf{k} are null and proportional. \square

The set $\mu(\mathbf{T})$ is never a vector space. If this set consists of a single linearly independent element then it is clearly a one-dimensional semi-space formed by the positive multiples of that element. In the case of having two linearly independent elements, say $\vec{\mathbf{k}}_1$ and $\vec{\mathbf{k}}_2$, we deduce that $\mu(\mathbf{T}) = \{\lambda_1 \vec{\mathbf{k}}_1, \lambda_2 \vec{\mathbf{k}}_2\}$, $\lambda_1, \lambda_2 \in \mathbb{R}^+$ since the linear combination of two null vectors can never be null.

The idea of principal direction has been already used in the literature mainly to classify algebraically the electromagnetic and Weyl tensors in Relativity. It is well known [45, 7] that these principal directions are (in four dimensions) precisely the principal directions according to definition A.2 of the energy-momentum tensor of the electromagnetic field and of the Bel-Robinson tensor, respectively, being both these tensors elements of \mathcal{DP}^+ . See [46] for a recent study of the principal directions of more general elements of \mathcal{DP}^+ . If $\mathbf{T} \in \mathcal{DP}_2^+|_x$ the calculation of $\mu(\mathbf{T})$ is accomplished by means of the next lemma proven in PII:

Lemma A.1 *For any $\mathbf{T} \in \mathcal{DP}_2^+|_x$, $\vec{\mathbf{k}} \in \mu(\mathbf{T})$ iff $\vec{\mathbf{k}}$ is a null eigenvector of \mathbf{T} .* \square

As a corollary of this we can give a result for higher rank tensors.

Corollary A.1 *For $\mathbf{T} \in \mathcal{DP}_r^+|_x$, $r \geq 4$, $\vec{\mathbf{k}} \in \mu(\mathbf{T}) \Leftrightarrow \vec{\mathbf{k}}$ is a null eigenvector of $\overbrace{i_{\vec{\mathbf{k}}} \dots i_{\vec{\mathbf{k}}}}^{r-2} \mathbf{T}$.*

Proof : Just apply recursively proposition A.1 $r - 2$ times to end up with a rank-2 tensor and then use lemma A.1. \square

For a rank-2 causal tensor the number of its linearly independent principal directions is the dimension of the subspace $Span\{\mu(\mathbf{T})\}$ which is always an eigenspace of the tensor \mathbf{T} for one of its positive eigenvalues if \mathbf{T} is symmetric (otherwise the scalar product of *null* vectors corresponding to different eigenvalues would have to be zero which is impossible). To get a better picture of this we show next a table with the allowed Segre types for symmetric rank-2 causal tensors together with the principal directions of each type.

$\dim(Span\{\mu(\mathbf{T})\})$	Segre type
0	$[1, 1 \dots 1]$ and its spatial degeneracies
1 (degenerate type)	$[2 \ 1 \dots 1]$ and its degeneracies
2	$[(1, 1) 1 \dots 1]$ and its spatial degeneracies
.....
$n - 1$	$[(1, 11 \dots 1) 1]$
n	$[(1, 11 \dots 1)]$

Table A1. Possible Segre types for a rank-2 symmetric causal tensor. It is understood that the spatial degeneracies do not coincide with the timelike ones.

A justification of this result for four dimensions can be found in many textbooks (see e. g. [51, 33]) although its generalization for n dimensions is quite straightforward (see e.g. theorem 1 of [9]) and it shall not be presented here.

The eigenvalue associated to $Span\{\mu(\mathbf{T})\}$ has multiplicity $\dim Span\{\mu(\mathbf{T})\}$ (this is the degenerate eigenvalue with round brackets in the Segre notation of table A1). The so-called degenerate type possesses a single null eigenvector and is the only type

with no orthonormal basis formed by eigenvectors. Each rank-2 symmetric causal tensor has a canonical form according to its Segre type being these forms further constrained once the dominant property is imposed.

Proposition A.2 *A rank-2 symmetric tensor \mathbf{T} is an element of $\mathcal{DP}_2^+|_x$ if and only if its algebraic type is one of table A1 and its eigenvalues satisfy the following properties:*

- the eigenvalue λ_0 associated to the timelike part is greater than or equal to the absolute value of the remaining eigenvalues (spacelike eigenvalues).
- If \mathbf{T} is of Segre type $[2\ 1\ \dots\ 1]$ or its degeneracies then it can be written as $\mathbf{T} = \mathbf{T}_0 + \lambda \mathbf{k} \otimes \mathbf{k}$ with $\mathbf{T}_0 \in \mathcal{DP}_2^+$ a nondegenerate type other than $[1, 1\ \dots\ 1]$ and \mathbf{k} the single null eigenvector of \mathbf{T} (which is a null eigenvector of \mathbf{T}_0 as well). The conditions are now those necessary for \mathbf{T}_0 plus $\lambda > 0$.

Proof : The possible algebraic types have been already discussed and the first statement of this proposition is a particular case of a more general result proven in lemma 4.1 of [49]. The second statement is needed in the proof of theorem 4.1 of PI and its proof is supplied there (see also theorem 2 of [9]). \square

The set of null eigenvectors can also be used to write the rank-2 symmetric causal tensors in a certain canonical form which is used in this work and was proven in PI.

Theorem A.1 *Every symmetric $\mathbf{T} \in \mathcal{DP}_2^+|_x$ can be written canonically as the sum $\mathbf{T} = \sum_{r=1}^n \mathbf{T}\{\Omega_{[r]}\}$ of rank-2 “super-energy tensors” $\mathbf{T}\{\Omega_{[r]}\} \in \mathcal{DP}_2^+|_x$ of simple r -forms $\Omega_{[r]}$. Furthermore, the decomposition is characterized by the null eigenvectors of \mathbf{T} as follows: if \mathbf{T} has m linearly independent null eigenvectors $\vec{\mathbf{k}}_1, \dots, \vec{\mathbf{k}}_m$ then the sum starts at $r = m$ with $\Omega_{[m]} = \mathbf{k}_1 \wedge \dots \wedge \mathbf{k}_m$ and the remaining simple forms $\Omega_{[r]}$ contain $\Omega_{[m]}$ as a factor; if \mathbf{T} has no null eigenvector then the sum starts at $r = 1$ with $\vec{\Omega}_{[1]}$ the timelike eigenvector of \mathbf{T} and again $\Omega_{[1]}$ is a factor of all the simple forms $\Omega_{[r]}$. \square*

Let us recall that the superenergy tensor of a r -form Σ is given by \ddagger :

$$T_{ab}\{\Sigma\} = \frac{(-1)^{r-1}}{(r-1)!} \left[\Sigma_{aa_2\dots a_r} \Sigma_b^{a_2\dots a_r} - \frac{1}{2r} g_{ab} \Sigma_{a_1\dots a_n} \Sigma^{a_1\dots a_n} \right]. \quad (\text{A.1})$$

If the form Σ is simple, i. e., it is the wedge product of r 1-forms $\mathbf{u}_1, \dots, \mathbf{u}_r$, then the tensor T_b^a is proportional to an involutory Lorentz transformation hence $T_{ab}T_c^b = \rho g_{ac}$ and all the vectors $\{\vec{\mathbf{u}}_k\}_{k=1,\dots,r}$ are eigenvectors of T_b^a with the same eigenvalue (PI). Indeed, the tensor T_b^a has only one or two different eigenvalues depending on whether ρ vanishes or not. In the non-vanishing case T_b^a can be brought into the diagonal form with $\text{Span}\{\vec{\mathbf{u}}_1, \dots, \vec{\mathbf{u}}_r\} = \text{Span}\{\mu(\mathbf{T})\}$ and its orthogonal space as the only eigenspaces of the two different eigenvalues whereas for ρ vanishing T_b^a can never be diagonalized and all the eigenvectors have 0 as eigenvalue (see next paragraph). Every null vector entering as a factor in a simple Σ is an element of $\mu(\mathbf{T}\{\Sigma\})$ so in the previous theorem $\mu(\mathbf{T})$ is always a subset of the set of principal directions of each term of the decomposition. An account of this and other properties of these tensors can be found in PI.

The simple r -form Σ can be normalized with $\Sigma_{a_1\dots a_r} \Sigma^{a_1\dots a_r} = (-1)^{r-1} 2r!$ so as to get an involutory Lorentz transformation $S_b^a = T_b^a\{\Sigma\}$ as its superenergy tensor

\ddagger Clearly $\mathbf{T}\{\Sigma\}$ is a generalization to arbitrary degree forms of the energy-momentum tensor of the electromagnetic field in terms of the Faraday 2-form \mathbf{F} . This generalizing idea can be pursued further and one can define superenergy tensors of more general objects than antisymmetric tensors [49, 46].

provided that Σ is not null. If Σ is null, it takes the form $\Sigma = \mathbf{k} \wedge \mathbf{v}_1 \dots \wedge \mathbf{v}_{r-1}$ where $\mathbf{v}_1, \dots, \mathbf{v}_{r-1}$ are spacelike vectors and \mathbf{k} is null (this is the case with $\rho = 0$), so that $\mathbf{T}\{\Sigma\} = f\mathbf{k} \otimes \mathbf{k}$. Involutory Lorentz transformations shall be referred to in this work as *Lorentz tensors* and they are always the superenergy tensor of a certain simple form as was explicitly proven in PI (theorem 6.1) so there exists a surjective map between Lorentz tensors and appropriate normalized simple forms. This relation is not injective though, as the simple normalized forms $\pm\Omega$ and $\pm^*\Omega$ give rise to the same Lorentz tensor $\mathbf{T}\{\Omega\}$, where the star stands for the Hodge dual. Therefore, we deduce that each Lorentz tensor has, up to duality and sign, an associated simple form given by (A.1).

If Σ is a simple r -form as the one defined above, then $^*\Sigma$ is also simple (a form is simple iff its dual is also simple), and formed by the wedge product of $n-r$ 1-forms which are a basis of $\perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ if $\Sigma \cdot \Sigma \neq 0$ or $\perp \text{Span}\{\mathbf{k}, \mathbf{v}_1, \dots, \mathbf{v}_{r-1}\}$ for the $\rho = 0$ case. Hence a superenergy rank-2 tensor can always be viewed as the superenergy of a r -form or the superenergy of the dual $(n-r)$ -form and every property stated in terms of the r -form admits a dual description stemming from the dual $(n-r)$ -form.

If $\rho \neq 0$ we can get a Lorentz tensor S_b^a as already explained. From this we can define the tensor $P_{ab} = \frac{1}{2}(g_{ab} + S_{ab})$ which will be useful in the main text. Straightforward properties of this tensor are $P_{ac}P_b^c = P_{ab} = P_{(ab)}$, $P_a^a = r$ and $\det(P_{ab})=0$ if $S_{ab} \neq g_{ab}$ from what we conclude that P_b^a is an orthogonal projector. The next result identifies the subspace to which it projects.

Proposition A.3 P_b^a is an orthogonal projector onto $\text{Span}\{\mu(\mathbf{S})\}$.

Proof : As we explained before, $\text{Span}\{\mu(\mathbf{S})\}$ and $\perp \text{Span}\{\mu(\mathbf{S})\}$ are the two eigenspaces associated to the only two eigenvalues of S_b^a which in this case are $+1$ and -1 respectively as it is obvious from the property $S_{ac}S_b^c = g_{ab}$. Since S_b^a can be diagonalized, the direct sum of the above two spaces must be equal to the entire vector space which means that by construction P_b^a has $\text{Span}\{\mu(\mathbf{S})\}$ as the eigenspace with $+1$ eigenvalue. \square

From the proof of this proposition, we deduce that either $\text{Span}\{\mu(\mathbf{S})\}$ or its orthogonal complement must be timelike. The fact that $S_{ab} \in \mathcal{DP}_2^+(V)$ implies that S_b^a maps future-directed vectors onto future-directed vectors so the timelike space is associated to the positive eigenvalue. Moreover, in view of the previous theorem we deduce that $\dim(\text{Span}\{\mu(\mathbf{T})\}) = n \Leftrightarrow \mathbf{T} \propto \mathbf{g}$ (this can also be seen from table A1). Therefore every rank-2 causal tensor with n linearly independent principal directions is up to a factor the metric tensor.

It is possible to enlarge the set of principal directions of a rank-2 causal tensor T_{ab} if we search for the set of vectors in $\partial\Theta^+|_x$ which are mapped to null vectors under the endomorphism T_b^a .

Definition A.3 The set of null directions of $\mathbf{T} \in \mathcal{DP}_2^+|_x$ is defined as follows:

$$\sigma(\mathbf{T}) = \{\vec{\mathbf{k}} \in \partial\Theta_x^+ : T_b^a k^b \in \partial\Theta_x^+\}$$

Clearly every pair k^a, l^a of future-pointing null vectors such that $T_{ab}k^ak^b = 0$ comply with the previous definition. Furthermore, if $l^a = T_b^ak^b$ for $k^b \in \sigma(\mathbf{T})$ then $l_a T_b^a = \lambda k_b$, $\lambda > 0$ as follows from the property $0 = l_al^a = T_{ab}l^ak^b$. Every principal direction of \mathbf{T} is also a null direction although the converse is not true in general. Actually $\sigma(\mathbf{T})$ can be worked out in terms of the principal directions of another tensor

related to \mathbf{T} (recall from PI that if M_{ab} and N_{ab} are tensors then their $1 - 1$ inner product is denoted as $(M \cdot_1 \times_1 N)_{ab} = M_{ca} N^c_b$. However, most of our applications will deal with symmetric causal tensors in which case we can drop the subindexes in the notation $\cdot_1 \times_1 = \cdot$, as in PI.)

Proposition A.4 *For every $\mathbf{T} \in \mathcal{DP}_2^+(V)$ we have that $\sigma(\mathbf{T}) = \mu(\mathbf{T} \cdot_1 \times_1 \mathbf{T})$*

Proof : Pick up any $\vec{\mathbf{k}} \in \sigma(\mathbf{T})$ and define the future directed null vector $n^a \equiv T^a_b k^b$. Then $0 = n_a n^a = T_{ab} k^b T^a_c k^c = (T \cdot_1 \times_1 T)_{bc} k^b k^c \Rightarrow \vec{\mathbf{k}} \in \mu(\mathbf{T} \cdot_1 \times_1 \mathbf{T})$. Conversely if $0 = (T \cdot_1 \times_1 T)_{ab} k^a k^b$ then, given that $\mathbf{T} \in \mathcal{DP}_2^+|_x$ we get that necessarily $T^a_b k^b$ must be future directed and null, proving thus that $\vec{\mathbf{k}} \in \sigma(\mathbf{T})$. \square

Another case of special relevance occurs when the set of principal directions is the same as the set of null directions. For instance we have

Proposition A.5 *Let \mathbf{T} of $\mathcal{DP}_2^+|_x$ be symmetric. Then*

- (i) *If \mathbf{T} is degenerate, then $\sigma(\mathbf{T}) = \mu(\mathbf{T})$ iff $T_0 \neq 0$, with T_0 as in proposition A.2.*
- (ii) *If \mathbf{T} has Lorentzian signature then $\sigma(\mathbf{T}) = \mu(\mathbf{T})$.*

Proof : For (i), we can write $\mathbf{T} = \mathbf{T}_0 + \lambda \mathbf{k} \otimes \mathbf{k}$, $\lambda > 0$, for a null \mathbf{k} with $\mu(\mathbf{T}) = \{\rho \vec{\mathbf{k}}\}$, $\rho > 0$. Therefore, for every $\vec{\mathbf{n}} \in \sigma(\mathbf{T})$ we have that the linear combination of future-directed vectors $T^a_b n^b + \lambda(k_b n^b)k^a$ must be null and future-directed, which implies that $T^a_b n^b \propto k^a$ or in other words $0 = T_{ab} k^a n^b$ and thus, if $\mathbf{T}_0 \neq 0$, $T_{ab} k^b \propto n_a$. But we know that $\vec{\mathbf{k}}$ is a null eigenvector of \mathbf{T}_0 with positive eigenvalue (see again proposition A.2), hence $\vec{\mathbf{k}} \propto \vec{\mathbf{n}}$ and we conclude that in this case $\sigma(\mathbf{T}) = \mu(\mathbf{T}) = \{\rho \vec{\mathbf{k}}\}$, $\rho > 0$. On the other hand, if $\mathbf{T}_0 = 0$ then $\sigma(\mathbf{T}) = \partial\Theta_x^+ \neq \mu(\mathbf{T})$ so the result follows. To prove (ii), observe first that if \mathbf{T} is degenerate and has Lorentzian signature, then necessarily $\mathbf{T}_0 \neq 0$ so this case is covered by (i) and we can concentrate only on non-degenerate \mathbf{T} . In this case, T^a_b has an orthonormal basis of eigenvectors and its signature is Lorentzian if and only if all the eigenvalues are positive. Thus, if we write \mathbf{T} in that orthonormal basis and compute $\mu(\mathbf{T} \times \mathbf{T})$ we get at once that $\text{Span}\{\mu(\mathbf{T})\} = \text{Span}\{\mu(\mathbf{T} \times \mathbf{T})\}$ which implies $\mu(\mathbf{T}) = \mu(\mathbf{T} \times \mathbf{T})$ and thus $\mu(\mathbf{T}) = \sigma(\mathbf{T})$. \square

Another result which involves the set of null directions and the set of principal directions is the following one.

Lemma A.2 *Let \mathbf{T}_1 and \mathbf{T}_2 be symmetric nondegenerate elements of $\mathcal{DP}_2^+|_x$ with the property $\sigma(\mathbf{T}_1) = \sigma(\mathbf{T}_2) = \mu(\mathbf{T}_1) = \mu(\mathbf{T}_2)$. Then we can find a positive constant α such that $\mathbf{T}_1 - \alpha \mathbf{T}_2 \in \mathcal{DP}_2^+$.*

Proof : We divide up in cases depending on the number of linearly independent principal directions. Suppose first that $2 \leq m = \dim(\text{Span}(\mu(\mathbf{T}_1))) = \dim(\text{Span}(\mu(\mathbf{T}_2)))$. Then $T_x(V) = \text{Span}(\mu(\mathbf{T}_1)) \oplus (\text{Span}(\mu(\mathbf{T}_1)))^\perp$ where due to the hypotheses, $\text{Span}(\mu(\mathbf{T}_1))^\perp$ is a $(n - m)$ -dimensional spacelike subspace of $T_x(V)$ which is invariant under the endomorphisms T^a_b and T^a_b so we can find an orthonormal basis in which \mathbf{T}_1 and \mathbf{T}_2 take the form

$$\mathbf{T}_1 = \begin{pmatrix} \mathbb{I}_{\lambda_1} & \\ & \mathbf{A}_1 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} \mathbb{I}_{\lambda_2} & \\ & \mathbf{A}_2 \end{pmatrix}. \quad (\text{A.2})$$

\mathbb{I}_{λ_1} and \mathbb{I}_{λ_2} are $m \times m$ square matrices given by $\mathbb{I}_{\lambda_1} = \text{diag}(\lambda_1, -\lambda_1, \dots, -\lambda_1)$, $\mathbb{I}_{\lambda_2} = \text{diag}(\lambda_2, -\lambda_2, \dots, -\lambda_2)$ whereas \mathbf{A}_1 and \mathbf{A}_2 are *symmetric* bilinear forms acting on a vector space equipped with an Euclidean scalar product. This means that we

can find an orthonormal basis with respect to this scalar product which diagonalizes both \mathbf{A}_1 and \mathbf{A}_2 from what we conclude that there exists an orthonormal basis for the Lorentzian scalar product bringing both \mathbf{T}_1 and \mathbf{T}_2 into their canonical diagonal form. Thus in this new basis we get the above written expressions for \mathbf{T}_1 and \mathbf{T}_2 where now $\mathbf{A}_1 = \text{diag}(a_1, \dots, a_{n-m})$ and $\mathbf{A}_2 = \text{diag}(b_1, \dots, b_{n-m})$ with the additional property of $\lambda_1 \geq \max\{|a_1|, \dots, |a_{n-m}|\}$ and its counterpart for λ_2 , \mathbf{A}_2 , which are the requirements for \mathbf{T}_1 and \mathbf{T}_2 to be in $\mathcal{DP}_2^+|_x$ (see proposition A.2). Now we must find a positive parameter α such that $\mathbf{T}_1 - \alpha\mathbf{T}_2$ is in $\mathcal{DP}_2^+|_x$ which will be achieved if we take α fulfilling the inequalities

$$\lambda_1 - \alpha\lambda_2 \geq |a_j - \alpha b_j|, \quad j = 1, \dots, n-m. \quad (\text{A.3})$$

(A.3) can be rewritten as:

$$\alpha\lambda_2 - \lambda_1 \leq a_j - \alpha b_j \leq \lambda_1 - \alpha\lambda_2 \Rightarrow \begin{cases} \alpha(\lambda_2 + b_j) \leq \lambda_1 + a_j \\ \alpha(\lambda_2 - b_j) \leq \lambda_1 - a_j \end{cases}. \quad (\text{A.4})$$

Here $\lambda_1 + a_j > 0 < \lambda_2 + b_j$ (if they were zero, then $\dim(\text{Span}\{\mu(\mathbf{T}_1)\})$ would be greater than m), and $\lambda_1 - a_j = 0 \Leftrightarrow \lambda_2 - b_j = 0$ due to the property $\sigma(\mathbf{T}_1) = \sigma(\mathbf{T}_2)$ (to see this just compute $\mu(\mathbf{T}_1 \times \mathbf{T}_1)$ and $\mu(\mathbf{T}_2 \times \mathbf{T}_2)$ in the above defined orthonormal basis) so we can always choose a positive α fulfilling both inequalities (A.4).

Suppose now that we are in the case with no principal directions. This means under our hypotheses that $\sigma(\mathbf{T}_1) = \sigma(\mathbf{T}_2) = \emptyset$ from what we conclude that the functions $f_1(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) = \mathbf{T}_1(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2)$ and $f_2(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) = \mathbf{T}_2(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2)$ are nowhere vanishing on $\partial\Theta^+|_x \times \partial\Theta^+|_x$. Define the set $\mathcal{S}^+ \subset \partial\Theta_x^+$ as follows (see [45]): choose Minkowskian coordinates $\{x^0, \dots, x^{n-1}\}$ on the vector space $T_x(V)$ and let $\vec{\mathbf{k}} \in \partial\Theta_x^+$ be in \mathcal{S}^+ if $x^0 = 1$ when expressed in these coordinates. This implies that for such $\vec{\mathbf{k}}$ its coordinates fulfill the constraint $1 = (x^1)^2 + \dots + (x^{n-1})^2$ and that every null vector $\vec{\mathbf{n}} \in \partial\Theta_x^+$ can be written as $\vec{\mathbf{n}} = \beta(\vec{\mathbf{n}})\vec{\mathbf{k}}$ for some $\vec{\mathbf{k}} \in \mathcal{S}^+$, $\beta(\vec{\mathbf{n}}) > 0$. Hence \mathcal{S}^+ is a compact set of $T_x(V)$ which in turn means that it must also be compact in Θ_x^+ since $\overline{\Theta_x^+} = \Theta_x^+$. Therefore the continuous functions f_1 and f_2 achieve positive upper and lower bounds on $\mathcal{S}^+ \times \mathcal{S}^+$, termed as M_1 and m_1 , M_2 and m_2 , respectively. According to this, the inequality $f_1(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) - \alpha f_2(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) \geq m_1 - \alpha M_2$ holding for every $\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2 \in \mathcal{S}^+$ tells us that we can choose $0 < \alpha \leq m_1/M_2$ such that $\mathbf{T}_1(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) - \alpha\mathbf{T}_2(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) \geq 0$ from what we deduce that this will also hold for every pair of null vectors in Θ_x^+ as is clear by just rewriting them in terms of their \mathcal{S}^+ partners. This proves that $\mathbf{T}_1 - \alpha\mathbf{T}_2 \in \mathcal{DP}_2^+|_x$ for such values of α as required. \square

Remark. Note that this proof can be repeated interchanging the roles of \mathbf{T}_1 and \mathbf{T}_2 and so we deduce that there exist a positive constant β such that $\mathbf{T}_2 - \beta\mathbf{T}_1 \in \mathcal{DP}_2^+|_x$.

Our last algebraic result relates the strict *null convergence condition* for a rank-2 symmetric tensor with the dominant energy condition under certain assumptions (recall that the tensor $\mathbf{T} \in T_2^0(x)$ fulfills the null convergence condition if $\mathbf{T}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \geq 0$ for every null vector $\vec{\mathbf{k}}$).

Lemma A.3 $\mathbf{T} \in T_2^0(x)$ satisfies $\mathbf{T}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) > 0 \ \forall \vec{\mathbf{k}} \in \partial\Theta_x$ if and only if there is a positive b such that $\mathbf{T}_{sym} + b\mathbf{g}|_x \in \mathcal{DP}_2^+|_x$ where \mathbf{T}_{sym} is the symmetric part of \mathbf{T} .

Proof : The implication from right to left is obvious so we will assume that \mathbf{T}_{sym} satisfies the strict null convergence condition but is not a future tensor (otherwise the result is obvious as well). We must prove that, for every null future-directed vector k^a , $(T_{(ac)} + b g_{ac})k^c$ is future-directed, or equivalently that $(\mathbf{T}_{sym} \times \mathbf{T}_{sym})(\vec{\mathbf{k}}, \vec{\mathbf{k}}) +$

$2b\mathbf{T}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \geq 0 \ \forall \vec{\mathbf{k}} \in \partial\Theta_x^+$. Consider the *compact* set \mathcal{S}^+ as in lemma A.2 and let m_1 and $m_2 > 0$ be the minima that $\mathbf{T}_{sym} \times \mathbf{T}_{sym}$ and \mathbf{T} reach over $\mathcal{S}^+ \times \mathcal{S}^+$, respectively. Then $(\mathbf{T}_{sym} \times \mathbf{T}_{sym})(\vec{\mathbf{k}}, \vec{\mathbf{k}}) + 2b\mathbf{T}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \geq m_1 + 2bm_2$ so that it is enough to choose $2b \geq -m_1/m_2$. Having proven the result on $\mathcal{S}^+ \times \mathcal{S}^+$ the assertion follows for every null future-directed vector $\vec{\mathbf{k}}$ by the same procedure as in lemma A.2. \square

Remark. When \mathbf{T} satisfies the null convergence condition and there are some null $\vec{\mathbf{k}}$ such that $\mathbf{T}(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = 0$ the result also holds requiring that $\mathbf{T}_{sym}(\vec{\mathbf{k}}, \cdot)$ be causal for these $\vec{\mathbf{k}}$, which is equivalent to demanding that $\vec{\mathbf{k}}$ is a null eigenvector of \mathbf{T}_{sym} .

In this work we deal with smooth sections of the set $\mathcal{DP}_r^+(V)$ and so we cannot expect that these sections keep the same algebraic character at all points of the manifold V . This becomes apparent if we work with rank-2 tensors so that one can use the tools already developed for the classification of other well known rank-2 tensors in General Relativity when they are seen as smooth sections of $T_2(V)$ [50, 24]. We specially quote the results of [50] which fit perfectly well in our scheme. Basically the authors show that one can decompose the manifold as the union of disjoint open sets (or sets with non empty interior) where any symmetric tensor has a constant Segre type plus a set with empty interior (see main text for a further explanation of this).

A.1. Causal relationship.

Let us consider two diffeomorphic n -dimensional Lorentzian manifolds (V, \mathbf{g}) and $(W, \tilde{\mathbf{g}})$. We will next introduce the following notions and properties taken from PII.

Definition A.4 A diffeomorphism $\varphi : V \rightarrow W$ is said to be a causal relation from V to W , denoted as $V \prec_\varphi W$, if $\varphi'\vec{\mathbf{u}} \in \Theta^+(W)$ for every $\vec{\mathbf{u}} \in \Theta^+(V)$. The Lorentzian manifold W is causally related with V , which is denoted simply as $V \prec W$, if such a diffeomorphism φ exists.

The most simple example of causal relation is a conformal mapping from a Lorentzian manifold to another. Here we remove the null cone preservation carried out by conformal transformations and just ask for the mapping of the whole Lorentzian cone Θ_x^+ into $\Theta_{\varphi(x)}^+$ for each point of the domain manifold V . It is also possible to define in the same fashion diffeomorphisms which map Θ_x^+ into $\Theta_{\varphi(x)}^-$ (*anticausal relations*). They can be brought into causal relations just by choosing the opposite causal orientation for the target manifold so we will only consider causal relations in this work.

Causal relations have the following interesting properties shown in PII:

Proposition A.6 The following statements are equivalent for two diffeomorphic Lorentzian manifolds (V, \mathbf{g}) and $(W, \tilde{\mathbf{g}})$.

- (i) $V \prec_\varphi W$
- (ii) $\varphi^*(\mathcal{DP}_r^+(W)) \subseteq \mathcal{DP}_r^+(V) \ \forall r \in \mathbb{N}$
- (iii) $\varphi^*(\mathcal{DP}_r^+(W)) \subseteq \mathcal{DP}_r^+(V)$ for a given odd $r \in \mathbb{N}$.

In addition to this \prec is a transitive relation i.e. $U \prec V$ and $V \prec W$ implies $U \prec W$. Nonetheless it is not an antisymmetric relation. \square

Although this proposition states important properties of causal relations it fails to give a practical procedure to find out if a given diffeomorphism is a causal relation. Next theorem from PII fills up this gap:

Theorem A.2 *A diffeomorphism $\varphi : V \rightarrow W$ satisfies $\varphi^* \tilde{\mathbf{g}} \in \mathcal{DP}_2^+(V)$ if and only if φ is either a causal or an anticausal relation.* \square

This result is quite valuable in the study of causal relations and causal symmetries because we can use the results dealing with causal tensors to study and classify these transformations. Nonetheless it is also possible to give another equivalent condition upon the tensor $\varphi^* \mathbf{g}$ which will turn out to be very useful as well.

Theorem A.3 *The diffeomorphism $\varphi : V \rightarrow W$ is either a causal or an anticausal relation iff $\varphi^* \tilde{\mathbf{g}}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) \geq 0, \forall \vec{\mathbf{k}} \in \partial\Theta_x$ and for every $x \in V$.*

Proof : One implication is obvious, so assume that $0 \leq \varphi^* \tilde{\mathbf{g}}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}}) = \tilde{\mathbf{g}}|_{\varphi(x)}(\varphi' \vec{\mathbf{k}}, \varphi' \vec{\mathbf{k}})$ for all null $\vec{\mathbf{k}}$. This tells us that $\varphi' \vec{\mathbf{k}}$ is causal for every null $\vec{\mathbf{k}}$ so we must only show that the causal orientation is consistently maintained for all them. Pick a pair of null vectors $\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2 \in \Theta_x^+(V)$ such that $\tilde{\mathbf{g}}|_{\varphi(x)}(\varphi' \vec{\mathbf{k}}_1, \varphi' \vec{\mathbf{k}}_1) > 0$ (if there is no such $\vec{\mathbf{k}}_1$ then every null vector goes to a null vector so that φ is actually a conformal relation, see theorem A.4 below) and assume that $\varphi' \vec{\mathbf{k}}_1 \in \Theta_{\varphi(x)}^+(W), \varphi' \vec{\mathbf{k}}_2 \in \Theta_{\varphi(x)}^-(W)$. This would mean $\tilde{\mathbf{g}}|_{\varphi(x)}(\varphi' \vec{\mathbf{k}}_1, \varphi' \vec{\mathbf{k}}_2) < 0$ so if we construct a continuous family of nonvanishing null vectors $\vec{\mathbf{n}}_\lambda \in \Theta_x^+(V), \lambda \in [0, 1]$ with the properties $\vec{\mathbf{n}}_0 = \vec{\mathbf{k}}_1, \vec{\mathbf{n}}_1 = \vec{\mathbf{k}}_2$ and $\vec{\mathbf{n}}_{\lambda_1} \neq \vec{\mathbf{n}}_{\lambda_2} \forall \lambda_1 \neq \lambda_2$ the function $f(\lambda) = \tilde{\mathbf{g}}|_{\varphi(x)}(\varphi' \vec{\mathbf{k}}_1, \varphi' \vec{\mathbf{n}}_\lambda)$ should vanish at a certain $\bar{\lambda} \in (0, 1)$. This would imply that $\varphi' \vec{\mathbf{k}}_1$ and $\varphi' \vec{\mathbf{n}}_{\bar{\lambda}}$ are null and proportional to each other, which is impossible as $\vec{\mathbf{k}}_1$ and $\vec{\mathbf{n}}_{\bar{\lambda}}$ are not proportional by hypothesis. \square

Since \prec is not a symmetric relation we conclude that it is a *preorder* in the set of Lorentzian manifolds. We may also have the possibility of W being causally related with V but not the other way round (denoted as $W \not\prec V$) which happens for instance if certain global causal properties holding for one of the manifolds does not for the other. This leads to

Definition A.5 *Two diffeomorphic Lorentzian manifolds (V, \mathbf{g}) and $(W, \tilde{\mathbf{g}})$ are called isocausal, written as $V \sim W$, if $V \prec W$ and $W \prec V$.*

According to this definition we can gather Lorentzian manifolds in equivalence classes defined by the equivalence relation $V \sim W \Leftrightarrow V \prec W$ and $W \prec V$. Each equivalence class $\text{coset}(V)$ can be thought of as a causal structure on the underlying manifold M . Indeed it is possible to define a partial order \preceq in $\text{Lor}(M)/\sim$ by

$$\text{coset}(V) \preceq \text{coset}(W) \Leftrightarrow V \prec W,$$

thus generalizing the well known hierarchy of causality conditions used in General Relativity. The relevance of \sim and \preceq in causality theory was addressed in PII.

A first classification of the causal relations can be performed according to the set of null vectors which remain null under its push-forward at a fixed point x (called *canonical null directions* of the transformation at x). This set, denoted as $\mu(\varphi)|_x$ for each causal relation φ , constitutes the part of the null cone preserved by φ at x so we get a gradation starting at the case where the full null cone is preserved and ending in the case without canonical null directions with a range of situations lying in between. The next result (see PII) permits us to calculate $\mu(\varphi)|_x$.

Proposition A.7 *The set $\mu(\varphi)|_x$ of canonical null directions of a causal relation $\varphi : V \rightarrow W$ at x is given by $\mu(\varphi^* \tilde{\mathbf{g}}|_x)$.*

Proof : The equality $\tilde{\mathbf{g}}|_{\varphi(x)}(\varphi'\vec{\mathbf{k}}, \varphi'\vec{\mathbf{k}}) = \varphi^*\tilde{\mathbf{g}}|_x(\vec{\mathbf{k}}, \vec{\mathbf{k}})$ implies that if a given null vector $\vec{\mathbf{k}} \in \Theta_x^+$ goes to a null vector $\varphi'\vec{\mathbf{k}}$ then $\vec{\mathbf{k}} \in \mu(\varphi^*\tilde{\mathbf{g}}|_x)$ which proves the inclusion $\mu(\varphi)|_x \subseteq \mu(\varphi^*\tilde{\mathbf{g}}|_x)$. The same equality serves also to prove the other inclusion. \square

With this result at hand we conclude that the same remarks made for the set of principal directions of a causal tensor hold for the canonical null directions of a causal relation at a point x . These local considerations can also be turned into global ones if we restrict our study to regions of the manifold in which the algebraic type of $\varphi^*\tilde{\mathbf{g}}$ does not change. Therefore, when we speak about the canonical null directions of a causal relation we will implicitly assume that the algebraic type of $\varphi^*\tilde{\mathbf{g}}$ does not change over the manifold for the transformation φ . Another important point is that we do not need to take $\sigma(\varphi^*\tilde{\mathbf{g}}|_x)$ into account.

Proposition A.8 $\sigma(\varphi^*\tilde{\mathbf{g}}|_x) = \mu(\varphi^*\tilde{\mathbf{g}}|_x)$ for a causal relation φ .

Proof : We already know the inclusion $\mu(\varphi^*\tilde{\mathbf{g}}|_x) \subseteq \sigma(\varphi^*\tilde{\mathbf{g}}|_x)$. If the couple of null vectors $\vec{\mathbf{k}}_1$ and $\vec{\mathbf{k}}_2$ are in $\sigma(\varphi^*\tilde{\mathbf{g}}|_x)$ then $0 = (\varphi^*\tilde{\mathbf{g}}|_x)(\vec{\mathbf{k}}_1, \vec{\mathbf{k}}_2) = \tilde{\mathbf{g}}|_{\varphi(x)}(\varphi'\vec{\mathbf{k}}_1, \varphi'\vec{\mathbf{k}}_2)$ which is only possible if $\varphi'\vec{\mathbf{k}}_1$ and $\varphi'\vec{\mathbf{k}}_2$, being both future-directed, are null and proportional meaning that $\vec{\mathbf{k}}_1$ (and $\vec{\mathbf{k}}_2$) are in $\mu(\varphi^*\tilde{\mathbf{g}}|_x)$ which leads to the other inclusion. \square

The transitivity of the causal relation \prec is due to the fact that the composition $\varphi_2 \circ \varphi_1$ of two causal relations $\varphi_1 : U \rightarrow V$ and $\varphi_2 : V \rightarrow W$ is also a causal relation. Nonetheless the inverse of a causal relation will in general fail to be a causal relation except in a very special case (see PII for the proof).

Theorem A.4 For a diffeomorphism $\varphi : V \rightarrow W$ the following properties are equivalent:

- (i) φ is a causal (or anticausal) relation with n linearly independent canonical null directions (i.e. every null vector goes to a null vector).
- (ii) $\varphi^*\tilde{\mathbf{g}} = \lambda\mathbf{g}$, $\lambda > 0$.
- (iii) $(\varphi^{-1})^*\mathbf{g} = \mu\tilde{\mathbf{g}}$, $\mu > 0$.
- (iv) φ and φ^{-1} are both causal (or anticausal) relations.

\square

Therefore we see that a conformal relation $\varphi : V \rightarrow W$ is characterized as a diffeomorphism such that $V \prec_\varphi W$ and $W \prec_{\varphi^{-1}} V$ from what we deduce that the only groups of causal relations are formed exclusively by conformal diffeomorphisms.

Another result of PII needed in this paper deals with how the causal character of causal vectors changes under a causal relation.

Proposition A.9 If $V \prec_\varphi W$ then:

- (i) $\vec{\mathbf{u}} \in \Theta^+(V)$ is timelike $\implies \varphi'\vec{\mathbf{u}} \in \Theta^+(W)$ is timelike.
- (ii) $\vec{\mathbf{u}} \in \Theta^+(V)$ and $\varphi'\vec{\mathbf{u}} \in \Theta^+(W)$ is null $\implies \vec{\mathbf{u}}$ is null.
- (iii) $\mathbf{k} \in \mathcal{DP}_1^+(W)$ is timelike $\implies \varphi^*\mathbf{k} \in \mathcal{DP}_1^+(V)$ is timelike.
- (iv) $\mathbf{k} \in \mathcal{DP}_1^+(W)$ and $\varphi^*\mathbf{k} \in \mathcal{DP}_1^+(V)$ is null $\implies \mathbf{k}$ is null.

\square

Appendix B

In this appendix we prove that every linear automorphism $\hat{\mathbf{T}} \in \mathcal{C}(\mathbb{L}, \boldsymbol{\eta})$ has at least an invariant causal direction, i.e there exists an element $\bar{\mathbf{u}}_x \in \Theta_x^+$ with the property $\hat{\mathbf{T}}\bar{\mathbf{u}}_x \propto \bar{\mathbf{u}}_x$. This result may well be available in other places of the literature under a different terminology although we have preferred to a proof adapted to our work (see for instance theorem 1.10.3 of [1] for an indirect proof in a special case). To start with, a preliminary result is needed [13].

Theorem B.1 (Brouwer's fixed point theorem) *Define $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and let $\varphi : B^n \rightarrow B^n$ be a continuous map. Then there exists a point $x_0 \in B^n$ such that $\varphi(x_0) = x_0$.*

Theorem B.2 *For every linear $\hat{\mathbf{T}} \in \mathcal{C}(\mathbb{L}, \boldsymbol{\eta})$ we can always find a vector $\bar{\mathbf{u}} \in \Theta_x^+$ such that $\hat{\mathbf{T}}(\bar{\mathbf{u}}) = \beta\bar{\mathbf{u}}$, $\beta > 0$.*

Proof : We start our proof by setting an equivalence relation in the vector space \mathbb{L} by means of the definition $\bar{\mathbf{u}}_1 \sim \bar{\mathbf{u}}_2 \Leftrightarrow \bar{\mathbf{u}}_1 = b\bar{\mathbf{u}}_2$ for some $b \in \mathbb{R}$. The elements of \mathbb{L}/\sim will be denoted with a square bracket enclosing a representative $\bar{\mathbf{u}}$ of each class as $[\bar{\mathbf{u}}]$. Clearly, it makes sense the definition $\hat{\Theta}_x = \{[\bar{\mathbf{u}}] \in \mathbb{L} : \bar{\mathbf{u}} \in \Theta_x\}$ as $b\bar{\mathbf{u}} \in \Theta_x \forall b \in \mathbb{R}$ if $\bar{\mathbf{u}} \in \Theta_x$. Now, define a system of Minkowskian coordinates $\{x^0, x^1, \dots, x^{n-1}\}$ in $(\mathbb{L}, \boldsymbol{\eta})$ and write every vector $\bar{\mathbf{u}} \in \mathbb{L}$ in these coordinates as $\bar{\mathbf{u}} = (a, ay^1, \dots, ay^{n-1})$, $a \in \mathbb{R}$. If $\bar{\mathbf{u}} \in \Theta_x$ then $(y^1)^2 + \dots + (y^{n-1})^2 \leq 1$ so it is possible to define a map $\hat{\Psi}_X : \hat{\Theta}_x \rightarrow B^{n-1}$ by $\hat{\Psi}_X([\bar{\mathbf{u}}]) \equiv (y^1, \dots, y^{n-1})$ where the subscript X means that this map depends on the chosen Minkowskian coordinate system. $\hat{\Psi}_X$ is a well-defined one-to-one map between $\hat{\Theta}_x$ and B^{n-1} which means that $\hat{\Psi}_X^{-1} : B^{n-1} \rightarrow \hat{\Theta}_x$ exists and is also one-to-one. Any $\hat{\mathbf{T}} \in \mathcal{C}(\mathbb{L}, \boldsymbol{\eta})$ is characterized by the properties of being linear and $\hat{\mathbf{T}}(\Theta_x^+) \subseteq \Theta_x^+$ so $\hat{\mathbf{T}}(\Theta_x) \subseteq \Theta_x$ and we can define the map $\hat{\varphi} : \hat{\Theta}_x \rightarrow \hat{\Theta}_x$ by $\hat{\varphi}([\bar{\mathbf{u}}]) \equiv [\hat{\mathbf{T}}(\bar{\mathbf{u}})]$, $\forall \bar{\mathbf{u}} \in \Theta_x$. Again this is a well-defined map and from it we construct another one given by $\hat{\Psi}_X \hat{\varphi} \hat{\Psi}_X^{-1} : B^{n-1} \rightarrow B^{n-1}$ which is continuous because it is the composition of continuous maps. According to Brouwer fixed point theorem, we know that there must exist a point $p \in B^{n-1}$ such that $\hat{\Psi}_X \hat{\varphi} \hat{\Psi}_X^{-1}(p) = p \Rightarrow \hat{\varphi} \hat{\Psi}_X^{-1}(p) = \hat{\Psi}_X^{-1}(p)$ which means that $[\bar{\mathbf{u}}_p] \equiv \hat{\Psi}_X^{-1}(p) \in \hat{\Theta}_x$ is a fixed point of $\hat{\varphi}$ as well. Therefore $\hat{\varphi}[\bar{\mathbf{u}}_p] = [\hat{\mathbf{T}}(\bar{\mathbf{u}}_p)] = [\bar{\mathbf{u}}_p] \Rightarrow \hat{\mathbf{T}}(\bar{\mathbf{u}}_p) = a_p \bar{\mathbf{u}}_p$ with $\bar{\mathbf{u}}_p \in \Theta_x$ and $a_p \in \mathbb{R}$. Given that the causal orientation of $\bar{\mathbf{u}}_p$ must be preserved under $\hat{\mathbf{T}}$ we conclude that a_p is a positive constant thus finishing the proof. \square

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